WZW-models in (2,2) Superspace

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for **Chris Hull's 60'th Birthday**

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A few remarks about Chris.

$(2,2)$ σ-model description of bihermitian geometry

Three types of $(2,2)$ superfields:

- chiral superfields $\phi, \bar{\phi}$ satisfying

$$\bar{D}_+\phi = 0, \quad \bar{D}_-\phi = 0,$$
$$D_+\bar{\phi} = 0, \quad D_-\bar{\phi} = 0,$$

- twisted chiral superfields $\chi, \bar{\chi}$ satisfying

$$\bar{D}_+\chi = 0, \quad D_-\chi = 0,$$
$$D_+\bar{\chi} = 0, \quad \bar{D}_-\bar{\chi} = 0,$$

- left and right semichiral superfields $\ell, \bar{\ell}, r, \bar{r}$ satisfying

$$\bar{D}_+\ell = 0, \quad \bar{D}_-r = 0,$$
$$D_+\bar{\ell} = 0, \quad D_-\bar{r} = 0.$$

Action in $(2,2)$ superspace

$$I = \int d^2\sigma\, d^4\theta\, K = \int d^2\sigma\, D^2 \bar{D}^2 K(\ell, \bar{\ell}, r, \bar{r}, \phi, \bar{\phi}, \chi, \bar{\chi})$$

The generalized Kähler potential K is defined modulo generalized Kähler transformations

$$K \mapsto K + f(\ell, \phi, \chi) + \bar{f}(\bar{\ell}, \bar{\phi}, \bar{\chi}) + g(r, \phi, \bar{\chi}) + \bar{g}(\bar{r}, \bar{\phi}, \chi),$$

Bihermitian structure

K fully encodes the local geometry of the target manifold, which consists of the structures (G, H, J_+, J_-)

- $J_\pm$ two integrable complex structures, g bihermitian:

$$J_\pm^2 = -\mathbb{1}, \quad N(J_\pm) = 0$$
$$G(J_\pm X, J_\pm Y) = G(X, Y),$$

- $H = d_+^c \omega_+ = -d_-^c \omega_-$, $dH = 0$, $\Leftrightarrow \nabla^{(\pm)} J_\pm = 0$ where $d_\pm^c$ are d^c operators with respect to $J_\pm$, and $\omega_\pm = GJ_\pm$ are hermitian 2-forms.

Bihermitian structure (discovered with Chris), is equivalent to generalized Kähler geometry.

Adapted coordinates

Three Poisson structures:

$$\pi_\pm = (J_+ \pm J_-)G^{-1} \ , \quad \sigma = [J_+, J_-]G^{-1}.$$

Superfields ϕ, χ, ℓ, r may be interpreted as coordinates adapted to these Poisson structures.

Note: ℓ, r coordinates involve a choice of polarization; K generates canonical transformations from holomorphic coordinates for J_+ to holomorphic coordinates for J_-:

$$J_+ \ : \ \phi, \chi, \ell, \tilde{\ell} := \frac{\partial K}{\partial \ell} \ , \ J_- \ : \ \phi, \bar{\chi}, r, \tilde{r} := \frac{\partial K}{\partial r} \ .$$

WZW-models

Maps from (super)-surface Σ to Lie group $\mathbf{G}$.
$(1,1)$ superspace action:

$$kI[g] = -\frac{k}{\pi}\int_\Sigma d^2\sigma\, d^2\theta\ \mathrm{tr}(g^{-1}\nabla_+ g g^{-1}\nabla_- g)$$

$$-\frac{k}{\pi}\int_B d^3\tilde\sigma\, d^2\theta\ \mathrm{tr}(\tilde g^{-1}\partial_t \tilde g\{\tilde g^{-1}\nabla_+\tilde g, \tilde g^{-1}\nabla_-\tilde g\})$$

k is an integer, $\Sigma = \partial B$, $\tilde g$ is an extension of g from Σ to B.

$(2, 2)$ WZW-models

A. Sevrin: all even dimensional reductive Lie groups super WZW-models have $2, 2$ supersymmetry.
On the Lie algebra, $\mathbb{J}_{\pm}$ are given by a choice of Cartan subalgebra and positive direction and both are $+i$ on the positive and $-i$ on the negative roots.
Since any two Cartan decompositions of a Lie algebra are related by group conjugation, the only freedom lies in the choice of the action on the Cartan subalgebra.

Type

Choice of $\mathbb{J}_+$ and $\mathbb{J}_-$ on the Lie algebra fixes the superfield content or type.

$$N_c = \dim_{\mathbb{C}} \ker(J_+ - J_-), \quad N_t = \dim_{\mathbb{C}} \ker(J_+ + J_-)$$

computed from $\ker(J_+ \pm J_-) \sim \ker(\mathbb{J}_+ \pm e_L e_R^{-1} \mathbb{J}_- e_R e_L^{-1})$, where $e_L e_R^{-1}$ is a group element. This gives $(\dim_{\mathbb{C}} \mathbf{G} - N_c - N_t)/2$ semichiral superfields.

Rank 2 groups

Group	N	$\mathbb{J}_+ = \mathbb{J}_-$			$\mathbb{J}_+ \neq \mathbb{J}_-$		
		N_s	N_c	N_t	N_s	N_c	N_t
$SU(2) \times U(1)$	2	1	0	0	0	1	1
$SU(2) \times SU(2)$	3	1	1	0	1	0	1
$SU(3)$	4	2	0	0	1	1	1
$SO(5)$	5	2	1	0	1	2	1
G_2	7	3	1	0	2	2	1

A remark on T-duality

Of the $\mathbf{G}_L \times \mathbf{G}_R$ Kac-Moody symmetries of the WZW model, only the Cartan torus subgroup $\mathbf{H}_L \times \mathbf{H}_R$ preserves both complex structures.

T-duality along a Kac-Moody isometry does not change the metric and torsion of a sigma model, but does change the type of the generalized geometry.

Thus, T-duality along a left (or right) isometry *relates the different generalized Kähler structures* on a given Lie group, and can be used to find the different generalized Kähler potentials.

Though we have known how to write WZW-models in (1,1) superspace for a long time, and understood bihermitian geometry in $(2,2)$ superspace for a long time, combining them has proved challenging.

The only cases understood before this work were the WZW-models on $SU(2) \times U(1)$ and $SU(2) \times SU(2)$.

The general case is still difficult, though the approach should be useful for many cases.

$SU(3)$ WZW-model

In the basis $h, \bar{h}, e_3, \bar{e}_3, e_1, \bar{e}_1, e_2, \bar{e}_2$, with

$$h = \begin{pmatrix} \frac{1}{2} + \frac{i}{2\sqrt{3}} & & \\ & -\frac{i}{\sqrt{3}} & \\ & & -\frac{1}{2} + \frac{i}{2\sqrt{3}} \end{pmatrix},$$

$$e_3 = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix}, \quad e_1 = \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}, \quad e_2 = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix},$$

$$\mathbb{J}_1 = \mathrm{diag}(i, -i, i, -i, i, -i, i, -i)$$

$$\mathbb{J}_2 = \mathrm{diag}(-i, i, i, -i, i, -i, i, -i).$$

Choices of complex structures

If we choose $J_{\pm}$ equal to each other at the origin, we find type (0,0) at generic points.

If we choose one to be $\mathbb{J}_1$ and the other $\mathbb{J}_2$, we find type (1,1).

Given $J_{\pm}$ at the origin, we find them anywhere by a left or right group action.

Since we know what the holomorphic/anti-holomorphic coordinates are at the origin, we also find them anywhere.

Type (0,0) coordinates for $SU(3)$

The $J_\pm$ holomorphic coordinates take the simplest form when presented in an overcomplete basis,

$$z_+^\phi = \log \bar{g}_{31}^{\bar{\omega}} g_{13}^{\omega}, \qquad z_+^\chi = \log \bar{g}_{11}^{\bar{\omega}} g_{33}^{\omega}, \qquad z_+^1 = \log \frac{g_{13}}{g_{23}},$$

$$z_+^2 = \log \frac{g_{23}}{g_{33}}, \qquad z_+^3 = \log \frac{\bar{g}_{11}}{\bar{g}_{21}}, \qquad z_+^4 = \log \frac{\bar{g}_{21}}{\bar{g}_{31}}$$

$$z_-^\phi = \log \bar{g}_{31}^{\omega} g_{13}^{\bar{\omega}}, \qquad z_-^{\bar{\chi}} = \log g_{11}^{\bar{\omega}} \bar{g}_{33}^{\omega}, \qquad z_-^1 = \log \frac{g_{11}}{g_{12}},$$

$$z_-^2 = \log \frac{g_{12}}{g_{13}}, \qquad z_-^3 = \log \frac{\bar{g}_{31}}{\bar{g}_{32}}, \qquad z_-^4 = \log \frac{\bar{g}_{32}}{\bar{g}_{33}},$$

Coordinate relations

where $\omega = e^{i\pi/3}$. These coordinates satisfy the relations

$$e^{z^1_\pm + z^3_\pm} + e^{-z^2_\pm - z^4_\pm} + 1 = 0$$

$$z^\phi_+ - z^\chi_+ = \omega(z^1_+ + z^2_+) - \bar\omega(z^3_+ + z^4_+)$$
$$z^\phi_- - z^{\bar\chi}_- = -\bar\omega(z^1_- + z^2_-) + \omega(z^3_- + z^4_-)$$

Choosing the semichiral coordinates

For the type $(0,0)$ generalized Kähler structure, $SU(3)$ is parametrized by two sets of semichiral coordinates. Then the Poisson structures are invertible at generic points.

One choice of Darboux semichiral coordinates σ:

$$\sigma(d\ell^j, d\tilde{\ell}^k) = \delta^{jk}\ ,\ \sigma(d\ell^j, d\ell^k) = \sigma(d\tilde{\ell}^j, d\tilde{\ell}^k) = 0$$

$$\sigma(dr^j, d\tilde{r}^k) = \delta^{jk}\ ,\ \sigma(dr^j, dr^k) = \sigma(d\tilde{r}^j, d\tilde{r}^k) = 0$$

is given by

Explicit coordinates

$$\ell^1 = \frac{1}{3}(z_+^\chi + 2z_+^\phi - \omega z_+^1 + \bar{\omega} z_+^4)$$

$$\tilde{\ell}^1 = (\bar{\omega} - \omega)(z_+^1 + z_+^2 + z_+^3 + z_+^4)$$

$$\ell^2 = z_+^\chi \quad , \qquad \tilde{\ell}^2 = z_+^\phi$$

$$\tilde{r}^1 = (\bar{\omega} - \omega)(z_-^1 + z_-^2 + z_-^3 + z_-^4)$$

$$r^1 = \frac{1}{3}(z_-^{\bar{\chi}} + 2z_-^\phi + \bar{\omega} z_-^2 - \omega z_-^3)$$

$$\tilde{r}^2 = z_-^{\bar{\chi}} \quad , \qquad r^2 = z_-^\phi$$

Many other choices are possible.

Integrability

To find the potential, we need to express the $\tilde{\ell}^j, \tilde{r}^j$ in terms of $\ell^j, \bar{\ell}^j, r^j, \bar{r}^j$.

The integrability condition

$$d(\theta_1 + \theta_2) = 0\ , \quad \theta_j := \tilde{\ell}^j d\ell^j + \bar{\tilde{\ell}}^j d\bar{\ell}^j - \tilde{r}^j dr^j - \bar{\tilde{r}}^j d\bar{r}^j \qquad \text{(no sum)}$$

guarantees the existence of a local potential $K(\ell^j, \bar{\ell}^j, r^j, \bar{r}^j)$ generating the symplectomorphism

$$\frac{\partial K}{\partial \ell^j} = \tilde{\ell}^j, \qquad \frac{\partial K}{\partial \bar{\ell}^j} = \bar{\tilde{\ell}}^j,$$
$$\frac{\partial K}{\partial r^j} = -\tilde{r}^j, \qquad \frac{\partial K}{\partial \bar{r}^j} = -\bar{\tilde{r}}^j.$$

The potential

K is given formally by

$$K = \int_{\mathcal{O}}^{(\ell^j,\bar{\ell}^j,r^j,\bar{r}^j)} \theta_1 + \theta_2,$$

where $\mathcal{O}$ is some base point. The integrability condition guarantees that the above expression is independent of integration path.

Final remarks

Once we find, e.g., action of the left Cartan torus on these coordinates (or any others related by different choice of polarization), we can use T-duality to find the K for type (1,1).

Open problems include:

- Extending this to the most general case.
- In the $SU(2) \times U(1)$ and $SU(2) \times SU(2)$ cases, dilogarithms arise; can we do the integrals for $SU(3)$ and higher cases?
- Understanding the case with $(4,4)$ supersymmetry.

Happy Birthday, Chris!