Non-relativistic holography

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Why non-relativistic holography?

- Gauge/gravity dualities have become an important new tool in extracting strong coupling physics.
- The best understood examples of such dualities involve relativistic (conformal) quantum field theories.
- Strongly coupled non-relativistic QFTs are common place in condensed matter physics and elsewhere.
- It is natural to wonder whether holography can be used to obtain new results about such non-relativistic strongly interacting systems.

String theorists' CMT motivations?

1. **Cold atoms at unitarity.**
   - Fermions in three spatial dimensions with interactions fine-tuned so that the *s*-wave scattering saturates the unitarity bound.
   - This system has been realized in the lab using trapped cold atoms [O'Hara et al (2002) ...].
   - It has been modeled theoretically by Schrödinger invariant theories with $z = 2$. [Son et al]
2. **High T_c superconductivity.**
   - Strange metal phases?

Plan

1. **Non-relativistic systems**
2. Schrödinger holography
3. Lifshitz holography
4. Conclusions and outlook

Holographic realization

Symmetries of the field theory should be realized holographically as isometries of the dual spacetimes.

Anti-de Sitter in $(D+1)$ dimensions admits as an isometry group the D-dimensional conformal group $SO(D,2)$.

Symmetries of a non-relativistic theory

In D spacetime dimensions the Galilean group consists of:

- temporal translations $\mathcal{H}$, spatial translations $\mathcal{P}^i$, rotations $\mathcal{M}^{ij}$ and Galilean boosts $\mathcal{K}^i$.

The Galilean algebra admits a central extension:

$$[K_i, P_j] = M\delta_{ij},$$

where M is the non-relativistic mass (or particle number).

Schrödinger symmetry group

The conformal extension adds to these generators:

- a dilation generator $\mathcal{D}_2$ and a special conformal generator $\mathcal{C}$.
- The dilatation symmetry $\mathcal{D}_2$ acts as

$$t \to \lambda^2 t, \qquad x^i \to \lambda x^i,$$

  i.e. with dynamical exponent $z = 2$.
- This is the maximal kinematical symmetry group of the free Schrödinger equation [Niederer (1972)], hence its name: Schrödinger group Sch_D.

Schrödinger with general exponent z

- One can also add to the Galilean generators (including the mass $\mathcal{M}$) a generator of dilatations $\mathcal{D}_z$ acting as

$$t \to \lambda^z t, \qquad x^i \to \lambda x^i$$

  but for general z there is no special conformal symmetry.
- This algebra will be denoted as $Sch_D(z)$.
- Removing the central term $\mathcal{M}$ gives the symmetries of a D-dimensional Lifshitz theory with exponent z, denoted $Lif_D(z)$.

Lifshitz spacetimes

The Lifshitz symmetry $Lif_D(z)$ may be realized geometrically in $(D+1)$ dimensions [Kachru et al, 2008]

$$ds^2 = \frac{dr^2}{r^2} - \frac{dt^2}{r^{2z}} + \frac{dx^i dx_i}{r^2}.$$

- As in AdS, the radial direction is associated with scale transformations: $r \to \lambda r$, $t \to \lambda^z t$, $x^i \to \lambda x^i$ is an isometry.

Holography for Schrödinger

[Son (2008)] and [K. Balasubramanian, McGreevy (2008)] initiated a discussion of holography for $(D+2)$ dimensional Schrödinger spacetimes,

$$ds^2 = -\frac{b^2 du^2}{r^4} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- This metric realizes geometrically the Schrödinger group with $z = 2$ in D dimensions: the radial direction is associated with dilatations, whilst another extra direction v is needed to realize the mass operator $\mathcal{M}$.
- In order for the mass operator $\mathcal{M}$ to have discrete eigenvalues the lightcone coordinate v must be compactified, giving a D-dimensional field theory with u the time coordinate.

Holography for general z Schrödinger

More generally one can also realize $Sch_D(z)$ geometrically in $(D+2)$ dimensions via

$$ds^2 = \frac{\sigma^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

- The dual field theory is then $(D+1)$-dimensional, with anisotropic scale invariance $u \to \lambda^z u$, $v \to \lambda^{2-z} v$ and $x^i \to \lambda x^i$.
- Various CMT models of this type e.g. Cardy's continuum limit of chiral Potts model in 2d ($z = 4/5$).
- The theory becomes a non-relativistic theory in D dimensions upon compactifying v or u.

Singularities and causal structure

By rescaling coordinates, AdS, Lifshitz and Schrödinger may all be written as

$$ds^2 = -b^2 \frac{dt^2}{r^{2z}} + \frac{1}{r^2}[dr^2 + dx^i dx^i + \eta dt dV].$$

- AdS is given by $b^2 = 0$: it has a coordinate horizon at $r \to \infty$.
- Lifshitz is given by $\eta = 0$: it has a null singularity at $r \to \infty$.
- Schrödinger also has a singularity as $r \to \infty$, and for $z > 1$ admits no global time function.
- We need $b^2 > 0$ for $z > 1$ for stability, and vice versa.

Basic questions about non-relativistic holography

What kind of strongly interacting Lifshitz and Schrödinger invariant theories can holography describe?

1. Matching of conductivities in specific phenomenological models to strange metal behavior?
2. Embedding into string theory and obtaining dualities from decoupling limits of brane systems will give much more information about specific Lifshitz or Schrödinger theory.

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Phenomenological models for Schrödinger

The $(D+2)$-dimensional Schrödinger spacetimes solve the field equations for Einstein gravity coupled to various types of matter. Simplest example [Son, 2008]:

- **Massive vector** model.

$$S = \int d^{D+2}x \sqrt{-G}[R - 2\Lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2 A_\mu A^\mu]$$

with $m^2 = z(D+z-1)$. Schrödinger metric supported by vector field with only a null component:

$$A_u = \frac{b}{r^z}.$$

Field theories dual to Schrödinger

- In general, the field theories dual to $(D+2)$-dimensional Schrödinger geometries [M.T. et al, 2010]

$$ds^2 = \frac{\sigma^2 du^2}{r^{2z}} + \frac{2dudv + dx^i dx^i + dr^2}{r^2},$$

can all be understood as Lorentz symmetry breaking deformations of $(D+1)$-dimensional CFTs e.g.

$$S_{CFT} \to S_{CFT} + \int dudvdx^i bX_v + \cdots$$

- Here X_v is a component of a vector operator, with relativistic dimension $(D+z)$.

Exactly marginal deformations

- For $z > 1$ this is an irrelevant deformation of the $(D+1)$-dimensional CFT, whilst for $z < 1$ it is relevant:

$$S_{CFT} \to S_{CFT} + \int dudvdx^i bX_v + \cdots$$

- Such deformations break the relativistic conformal symmetry but are **exactly marginal** with respect to $Sch_D(z)$ symmetry.
- A specific example of a $z = 4/5$ deformation of a 2d CFT was given in [Cardy, 1991] in the context of critical limits of the chiral Potts model.

Embedding into string theory

- Such massive vector solutions can be uplifted to 10d string theory backgrounds e.g. for $D = 3$ and $z = 2$ [Maldacena et al, Herzog et al, Adams et al 2008]

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}(2dudv - \frac{b^2}{r^2}du^2 + (dx^i)^2) + d\Omega^2_{S^5};$$

$$B_2 = \frac{b}{r^2}\eta \wedge du; \qquad F_5 = (d\Omega_{S^5} + *d\Omega_{S^5}),$$

  with η a certain Killing vector on S^5.
- Solutions can preserve supersymmetry, depending on the specific Killing vector of the S^5.
- Generalizations to finite temperature black brane solutions are also known.

Relation to branes

The dual field theory is a decoupling limit of D3-branes with a B_2 flux along a null worldvolume direction - resulting in a non-commutative dipole deformation of the CFT.

Dipole deformations

- To every field Φ is associated a dipole length L^{μ} (related to its global charge), and the non-commutative dipole product $*$ of two fields is [Ganor et al, 2000]

$$\Phi_1 * \Phi_2 = \Phi_1(x^{\mu} - L_2^{\mu})\Phi_2(x^{\mu} + L_1^{\mu}).$$

  From every "ordinary" field theory, a corresponding dipole field theory is obtained by using the dipole product.
- Expanding out the dipole product for null dipoles gives Schrödinger invariant deformations of the CFT e.g. for $\mathcal{N} = 4$ SYM

$$S_{SYM} \rightarrow S_{SYM} + \int d^4 x b \mathcal{V}_v + \mathcal{O}(b^2)$$

  where V_v is a dimension five vector operator, as above.
- Null dipole theories exhibit no (apparent) IR/UV mixing problems but are non-local in the v direction.

Discrete Lightcone Quantization (DLCQ)

- To obtain a non-relativistic system we still need to compactify the v direction (for $z > 1$) or the u direction (for $z < 1$).

But periodically identifying a null circle is subtle!

- The zero mode sector is usually problematic (and here the problem is seen in ambiguities in the initial value problem in the spacetime).
- Strings winding the null circle become very light.

Schrödinger phenomenology: a generic prediction

- In the bulk geometries the deformation parameter b can take any value.

The physical systems being modeled should have a corresponding parameter, adjusting which preserves the quantum criticality.

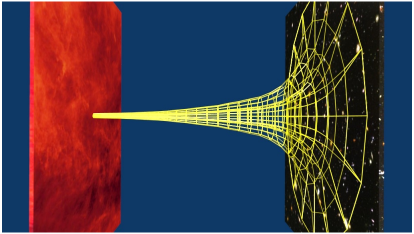
- How can we stabilize this modulus?

Schrödinger phenomenology

- Looking at the $z = 2$ metric

$$ds^2 = \frac{dr^2}{r^2} - b^2 \frac{du^2}{r^4} + \frac{2dudv + dx^i dx_i}{r^2}$$

recall that from the perspective of the original CFT, the deformation was irrelevant.



- Naively, the IR behavior is thus dominated by that of the original CFT, whilst the UV behavior is that of a $z = 2$ theory.
- Placing probe branes in the background, and computing the conductivities of charge carriers on these branes, one indeed sees such behavior. [Ammon et al, 2010]

Schrödinger summary

Geometric realization of the mass generator $\mathcal{M}$ of the Schrödinger algebras is undesirable, and leads to the dual theory being a DLCQ of a deformed CFT.

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with $m^2 = z(D+z-2)$. Lifshitz metrics are supported by a vector field with only a timelike component:

$$A_t = \frac{1}{r^z}.$$

Embedding into string theory

Surprisingly difficult to embed Lifshitz into string theory:

- Use higher derivative gravity, $R + R^2$ - useful for finding analytic black holes, but not a string theory embedding.
- Use gauged supergravities arising from e.g. reductions of massive type IIA. [Gregory et al]
- Embed into Sasaki-Einstein reductions (specific values of z, some are DLCQ of deformed CFTs). [Gauntlett et al]
- Brane and F theory constructions. [Hartnoll et al]

Lifshitz phenomenology

- Lifshitz spacetimes are always supported by matter such as massive vectors, or by higher derivative curvature terms.

Holography implies that matter or higher derivative gravity in the bulk is dual to operators in the Lifshitz theory.

- What is the physical role of these operators in the quantum critical theory in general?

Generic features of Lifshitz phenomenology

Even without a complete string theory embedding, we can explore generic features of the dual theories.

- For example, given (D, z) one can easily compute correlation functions of operators of dimension Δ at $T = 0$:

$$\langle \mathcal{O}(t, x^i)\mathcal{O}(0,0)\rangle = \frac{1}{(x^i x_i)^{\Delta}} \mathcal{F}_{\Delta}\left(\frac{x^i x_i}{t^{2/z}}\right)$$

  The functions $\mathcal{F}_{\Delta}(y)$ are determined by solutions to hypergeometric equations.

Now let's turn to transport properties...

Modified Lifshitz holography

- Consider an action which includes a gauge field and a scalar:

$$S = \int d^{D+1}x\sqrt{-G}[R - \frac{1}{2}(\partial\Phi)^2 + g(\Phi)F_{\mu\nu}F^{\mu\nu} + V(\Phi)]$$

  These actions admit Lifshitz black hole solutions

$$ds^2 \sim -f(r)r^{\beta}dt^2 + \frac{dr^2}{r^{\beta}f(r)} + r^{\gamma}dx^i dx_i,$$

  with $f(r) = 0$ at the horizon, and $f(r) = 1$ in zero temperature solutions. (Zero entropy extremal BH!)

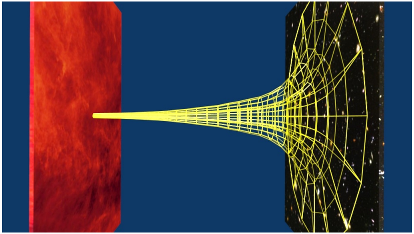
- The metric is Lifshitz at $T = 0$, but the field equations enforce a running scalar

$$\Phi \sim \log(r),$$

  which breaks the scale invariance.
- Many choices of the functions in the action can be embedded into string theory.

Modified Lifshitz and strange metals

Probe branes in modified Lifshitz can model charge carriers interacting with the quantum critical theory:



- The charge carriers have DC resistivity $\rho \sim T^{v_1}$ and AC conductivity behaves as $\sigma(\omega) \sim \omega^{-v_2}$, with nontrivial v_1 and v_2. [Hartnoll et al, 2009]
- $v_1 = v_2 = 2/z$ for pure Lifshitz so $z = 2$ reproduces strange metal behavior for DC conductivity and $z \sim 3$ for AC conductivity.
- Various values of (v_1, v_2) can be obtained in modified Lifshitz.

Lifshitz outlook

A better understanding of embedding (modified) Lifshitz into string theory is crucial to understand the key features of the quantum critical theory and allow models for strange metal behavior to be developed further.

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Non-relativistic holography: status

General success:

- Simple phenomenological models capture key features of strongly interacting non-relativistic theories.

Main problem:

- Neither Lifshitz nor Schrödinger has been satisfactorily embedded into top-down string theory models, and many holographic calculations are technically and conceptually challenging.

Non-relativistic bulk theory?

- Using relativistic (Einstein) gravity to model a non-relativistic field theory is perhaps counter-intuitive.

Should one instead take a non-relativistic limit of AdS/CFT, in which the bulk theory is Newtonian? [Bagchi, Gopakumar]

- It is not clear how one would model Lifshitz etc in the bulk and how finite temperature physics would work without black holes.

Future prospects?

Holographic models look promising for describing a wide range of non-relativistic strongly interacting systems.