Stationary holographic plasma quenches and numerical methods for non-Killing horizons

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We explore use of the harmonic Einstein equations to numerically find stationary black holes where the problem is posed on an ingoing slice that extends into the interior of the black hole. Requiring no boundary conditions at the horizon beyond smoothness of the metric, this method may be applied for horizons that are not Killing. As a non-trivial illustration we find black holes which, via AdS-CFT, describe a time-independent CFT plasma flowing through a static spacetime which asymptotes to Minkowski in the flow's past and future, with a varying spatial geometry in-between. These are the first non-perturbative examples of stationary black holes which do not have Killing horizons. When the CFT spacetime slowly varies, the CFT stress tensor derived from gravity is well described by viscous hydrodynamics. For fast variation it is not, and the solutions are stationary analogs of dynamical quenches, with the plasma being suddenly driven out of equilibrium. We find evidence these flows become unstable for sufficiently strong quenches, and speculate the instability may be turbulent.

Due to the remarkable Anti de Sitter-Conformal Field Theory (AdS-CFT) correspondence [1–3], the behaviour of black holes in asymptotically AdS spacetimes is equivalent to the behaviour of hot plasma in certain strongly coupled CFTs. Since these black holes may have planar horizons they admit arbitrarily long wavelength perturbations which give rise to the hydrodynamic behaviour expected of the CFT plasma [4–8]. Perturbations on short scales correspond to microscopic plasma behaviour beyond hydrodynamics. As this currently cannot be computed directly in strongly coupled CFTs, gravity provides an entirely new computational tool, as reviewed in [9–11]. This has been exploited for dynamical quenches where the CFT is abruptly perturbed [12–16] and the dual spacetime is found by numerical methods [17–23].

Here we study an analog of a dynamical quench, where the CFT state is time independent. We consider stationary black holes dual to a time independent relativistic plasma flow through a static spacetime. This asymptotes to Minkowski in the flow's past and future, but in-between the spatial geometry varies in the flow direction. The flow, initially in equilibrium, is forced out of equilibrium in response to passing through the curved spacetime region, before returning to equilibrium afterwards. For slowly varying spacetimes (with respect to the length scale set by the local temperature) such flows are well described by hydrodynamics [24]. For quick variation they probe behaviour beyond hydrodynamics, and are stationary analogs to dynamical quenches.

These black holes are of a qualitatively new variety, being the first non-perturbative examples of stationary black holes that do not have Killing horizons, and hence do not move rigidly [25]. The rigidity theorem states that if a stationary horizon is compact, it is also Killing [26–28]. Our black holes have non-compact horizons and evade this theorem, and since the dual plasma flows in a direction which is not a symmetry, these horizons are not Killing. Other stationary non-Killing horizons have been considered in the AdS-CFT context; 'flowing funnels' [29] where so far only related solutions with Killing horizons have been found [30–32], and 'plasma shocks' [33, 34] where non-Killing horizons have been found in a perturbative limit, but are expect to exist beyond this.

Harmonic Einstein equations.— Consider a Lorentzian stationary solution to the Einstein equations where the stationary Killing vector field T is globally timelike. We consider the purely gravitational case $R_{\mu\nu} = \Lambda g_{\mu\nu}$, although generalisation to include matter is straightforward. We adapt coordinates, $x^\mu = (t, x^i)$, so $T = \partial/\partial t$ and the metric $g_{\mu\nu}$ is,

$$ds^2 = -N(x)(dt + A_i(x)dx^i)^2 + b_{ij}(x)dx^i dx^j \,. \quad (1)$$

The spacetime is Lorentzian so $\det g_{\mu\nu} = -N \det b_{ij} < 0$, and T is globally timelike so $N(x) > 0$ and thus $b_{ij}(x)$ is a Riemannian metric. In order to obtain a well posed problem we must eliminate the coordinate invariance. Instead of solving the Einstein equations, we solve 'harmonic' or 'DeTurck' Einstein equations [35, 36] as reviewed in [37],

$$R^H_{\mu\nu} \equiv R_{\mu\nu} - \nabla_{(\mu}\xi_{\nu)} = \Lambda g_{\mu\nu} \quad (2)$$

where, $\xi^\mu = g^{\alpha\beta}\left(\Gamma^\mu{}_{\alpha\beta} - \bar{\Gamma}^\mu{}_{\alpha\beta}\right)$ is constructed from a fixed reference connection $\bar{\Gamma}^\mu{}_{\alpha\beta}$ on the manifold, which here we take to be the connection of a reference metric $\bar{g}_{\mu\nu}$. The two derivative part of these equations is governed by the operator $b^{ij}\partial_i\partial_j$, and since b_{ij} is a Riemannian metric the harmonic Einstein equation is elliptic.

For suitable boundary conditions the whole system may be solved as a standard elliptic boundary value problem. We want solutions with $\xi^\mu = 0$, which is a coordinate gauge condition analogous to generalised harmonic gauge in dynamical numerical GR [38], and must ensure our boundary conditions are compatible with this. In certain cases one may prove ξ^μ must vanish [30]. In general, 'soliton' solutions with $\xi^\mu \neq 0$ may exist. However for an elliptic problem solutions are locally unique, and hence one may easily distinguish whether a solution found has vanishing ξ^μ or not. The system may be solved by relaxation which is related to Ricci flow. Alternatively after discretization it can be solved by the Newton method given an initial guess.

Old method for Killing horizons.— For a stationary black hole $T = \partial/\partial t$ is not globally timelike, being spacelike inside the horizon, or outside if an ergoregion exists. Hence b_{ij} must become Lorentzian, and the problem inside the horizon and ergoregion naively appears hyperbolic.

A previous method [36] focussed on retaining ellipticity by assuming a Killing horizon that rigidly moves. We assume Killing vectors $R_a = \partial/\partial y^a$ exist which commute with themselves and T. For constants Ω^a we take $K = T + \Omega^a R_a$ to generate the Killing horizon and rigid motion of the spacetime. The metric can then be written as,

$$ds^2 = G_{AB}(x)(dy^A + A^A_a(x)dx^a)(dy^B + A^B_b(x)dx^b) + b_{ab}(x)dx^a dx^b \tag{3}$$

with $y^A = (t, y^a)$. Now G_{AB} is Lorentzian outside the horizon (even in an ergoregion), and degenerates at the horizon or axes of symmetry of the R_a. Correspondingly $b_{ab}(x)$ can be chosen to be Riemannian on, and in the exterior of, the horizon. These coordinates yield a slice of the spacetime that intersects the bifurcation surface of the Killing horizon. Since the principle part of R^H is governed by $b^{ab}\partial_a\partial_b$ the p.d.e. system is elliptic posed on the base geometry with coordinates x^a - the 'orbit space'. The boundaries of this base are where G_{AB} degenerates and smoothness determines boundary conditions there [36, 39, 40]. The surface gravity κ and (angular) velocities of the horizon Ω^a are prescribed in these boundary conditions. For regularity the reference metric must also have a Killing horizon at the same location with the same κ and Ω^a. Hence we may think of the reference metric as specifying these moduli of the solution.

New method for non-Killing horizons.— Suppose we are interested in stationary black holes that do not have a Killing horizon. For a non-Killing horizon we cannot assume existence of a bifurcation surface and regular past horizon. In the new approach we now describe we no longer require the problem to be elliptic. We take the general stationary ansatz (1) and pose the harmonic Einstein equations on an ingoing slice (analogous to that of Eddington-Finklestein) that intersects the future horizon and extends into the black hole interior. For the metric (1) in ingoing coordinates $g_{\mu\nu}$ is regular at the future horizon, so $\det g_{\mu\nu} = -N \det b_{ij} < 0$ on the future horizon and its exterior. Thus b_{ij} is elliptic in the exterior of the horizon or ergoregion (ie. where $N > 0$) and is hyperbolic inside these (where $N < 0$). Such a problem is analogous to mixed hyperbolic elliptic p.d.e.s in fluid dynamics. Whilst the problem will have hyperbolic character inside the horizon and ergoregion we may still solve it using the Newton method as before. Interestingly the Ricci flow method appears also to work, but we will not explore that here. The components of $\xi^\mu = 0$ give gauge conditions; ξ^i are associated to x^i coordinate freedom and ξ^t to the freedom $t \to t + f(x^i)$.

An important difference to the old method is that since the problem is hyperbolic in the interior of the horizon, at the innermost points of our domain we impose only the harmonic Einstein equations and no boundary condition. The requirement that the metric is smooth in our domain is sufficient to ensure regularity of the horizon in ingoing coordinates. Starting from a smooth initial guess near a solution, then the Ricci flow and Newton method will preserve smoothness, since both update the metric using the harmonic Ricci tensor which will also be smooth. We implicitly assume the physically reasonable statement that asymptotic boundary conditions together with future horizon regularity define a locally unique stationary black hole solution, up to moduli of the solution (such as mass). This is true for Killing horizons as can be seen from the elliptic nature of the p.d.e.s discussed earlier. Indeed the black hole uniqueness theorems show in many cases global uniqueness. For stationary non-Killing horizons we assume local uniqueness here, but emphasise we know of no proof. We note this is the basis of the fluid/gravity correspondence [7, 8]. It is the horizon rather than the ergosurface where smoothness must be imposed, even though the ergosurface determines the transition of character of the p.d.e.s. This is analogous to a stationary scalar field in the Kerr background, where one explicitly sees the scalar equation has regular singular behaviour at the horizon, and hence it is smoothness there that constrains the solutions [41].

A second key difference with the old method is that the reference metric, while selecting the coordinate system, no longer specifies the surface gravity or velocities of the horizon. These moduli must be fixed by appropriate boundary conditions. Indeed the horizons of the solution and reference metric in general do not coincide.

We provide a simple toy example in the Supplemental Material which implements the old and new methods; finding Schwarzschild taking static spherical symmetry.

Holographic plasma quenches.— We now use this method to find stationary black holes that are locally asymptotically AdS_4, and by AdS-CFT describe CFT stationary plasma flows in a non-trivial geometry. These are Einstein metrics solving $R_{\mu\nu} = -\frac{3}{l^2} g_{\mu\nu}$. We choose units so that the AdS length $l = 1$. These geometries have a conformal boundary we must specify which gives the CFT spacetime. We take,

$$ds^2 = -dt^2 + d\rho^2 + \sigma(\rho)dy^2 \tag{4}$$

which is static and inhomogeneous in ρ, with,

$$\sigma(\rho) = 1 + \frac{\alpha}{2}\Big(1 + \tanh(\beta\rho)\Big) \tag{5}$$

for constants α, β, so that the geometry asymptotes to Minkowski for $\rho \to \pm\infty$. We take the CFT plasma to be stationary, homogeneous in y, and flowing from $\rho = -\infty$ to $+\infty$. In the asymptotic regions the plasma flow becomes homogeneous, and the dual is a homogeneous black brane there. In between the plasma flows in a direction that is not an isometry; hence the plasma flow is inhomogeneous, and the dual black hole does not have a Killing horizon.

We take the ingoing plasma to be subsonic with velocity $v_0 < 1/\sqrt{2}$ and temperature T_0. Using the holographic fluid/gravity correspondence [7, 8, 42] provided the boundary metric gradients are sufficiently small, meaning $\beta/T_0 \to 0$, then the plasma behaves as an ideal fluid. The first deviation from ideal behaviour is due to shear viscosity. Upon increasing β/T_0 towards unity, one expects the derivative expansion of hydrodynamics to break down completely in the region $|\beta\rho| \sim O(1)$ where the boundary metric is highly curved, as microscopic physics is required to describe small scale plasma phenomena. Such solutions represent stationary flowing plasma quenches.

We write an ansatz for these metrics as,

$$ds^2 = \frac{1}{z^2}\Big(-Tdt^2 + 2Vdtdz + 2Udtd\rho + Adz^2 + B\left(d\rho + Fdz\right)^2 + Sdy^2\Big) \tag{6}$$

with the functions T, V, B, S, U, F, A being smooth (or at least C^2) in ρ and z. The locally AdS_4 boundary is at $z = 0$, and we impose the boundary conditions such that

$$T = V = A = B = 1 \; , \; U = F = 0 \; , \; S = \sigma(\rho) \tag{7}$$

there. Using holographic renormalisation [43] we identify the boundary metric (4), and the vev of the CFT stress tensor is given by third z derivatives of $T, V, \ldots, A$ at $z = 0$; details are given in the Supplemental Material.

For regularity at the locally AdS boundary we require the reference metric to obey the same boundary conditions as the metric. We choose the reference metric to be a boosted black brane with S deformed to obey (7);

$$\begin{gathered} S = \sigma(\rho)\,,\; T = 1 - c_r^2\left(z/z_0\right)^3\,,\; B = 1 + s_r^2\left(z/z_0\right)^3 \\ F = -s_r/B\,,\; A = -s_r^2/B\,,\; V = c_r\,,\; U = s_r c_r\left(z/z_0\right)^3 \end{gathered} \tag{8}$$

for constants z_0, r and $c_r = \cosh r$, $s_r = \sinh r$. We also use this for the initial guess. We compactify ρ as $d\rho = dx/(1-x^2)^2$ and work in the coordinate domain $x \in [-1, 1]$ and $z \in [0, z_{max}]$. Our solutions have a future horizon located at $z = H(x) < z_{max}$ within this domain so that horizon regularity is imposed by smoothness of $T, \ldots, A$. Since $\rho \sim 1/(x \mp 1)$ as $x \to \pm 1$ and black brane perturbations decay exponentially in ρ as $\rho \to \pm\infty$ we expect for our reference metric the functions $T, \ldots, A$ will have all x derivatives vanish at $x \to \pm 1$.

For $z = z_{max}$ and $-1 < x < -1$ we impose the equations of motion as for the interior points. At $x = \pm 1$ we impose Neumann boundary conditions. We expect two moduli, the ingoing surface gravity and velocity of the horizon, dual to T_0 and v_0. These moduli are not fixed by the reference metric in this ingoing method. To obtain a locally unique solution we fix two pieces of data. A numerically stable method is to fix Dirichlet data for V at the point $(z, x) = (z_{max}, -1)$ and T at $(z, x) = (z_{max}, +1)$ setting the values to those of the reference metric. Thus z_0 and r in the reference metric control the ingoing plasma data T_0 and v_0.

We use finite differencing and discuss the tests of convergence in detail in the Supplemental Material. Our code produces approximately fourth order convergence. For the resolutions used, up to 70×280 in z and x, the maximum fractional local error in the Einstein equations outside the horizon is better than $\sim 10^{-7}$. Hence these are very good numerical solutions. Convergence tests for extraction of the stress tensor (which depends on multiple derivatives) indicate better than percent level accuracy.

From hydrodynamics to quenches.— We now present data where the ingoing homogeneous plasma has subsonic velocity $v_0 = 0.50$, and temperature $T_0 = 0.24$ in our units. Since the boundary theory is a CFT, any other temperature is related by an appropriate scaling, and this value is taken for convenience. We choose the boundary metric to have $\alpha = 0.4$, and we adjust β to move between a slowly or rapidly varying geometry. This value of α is sufficiently large that the boundary metric deformation from Minkowski cannot be described by perturbation theory. As we shall see, the deviation from homogeneous behaviour will correspondingly be large. With these data we find the dual gravity solution and from it extract the vevs of the CFT stress tensor components $T_{tt}, T_{t\rho}, T_{\rho\rho}$ and T_{yy}. The conservation equation together with tracelessness implies that all the information in the stress tensor is characterised by a single function of ρ. We choose to plot the (scale invariant) function v defined by, $v/\left(1 + v^2\right) = \langle T^{t\rho}\rangle / \langle T^{tt} + T^{\rho\rho}\rangle$ for $0 \le v \le 1$. v gives the local velocity of the plasma in the stationary frame, ie. the velocity relative to the stationary frame required to boost into the rest frame of the plasma. We emphasise this does not depend on any hydrodynamic interpretation of the stress tensor. In figure 1 we plot this function for various β between 0.2 and 2. We show the same quantity for the fluid/gravity viscous hydrodynamics approximation (ie. the fluid velocity) with the same ingoing data - see the Supplemental Material for details.

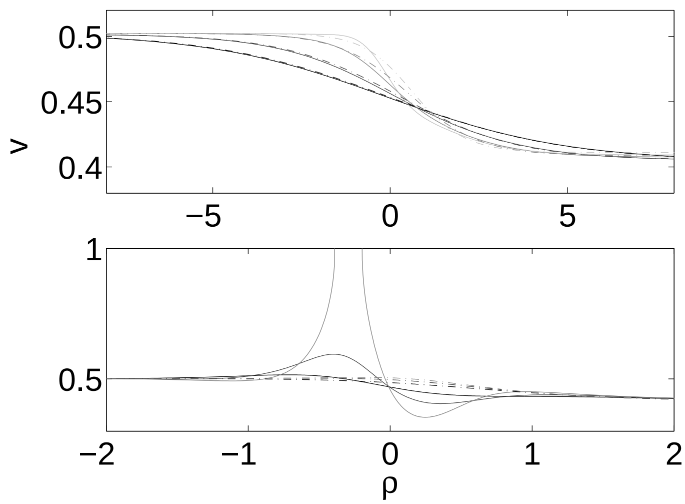

FIG. 1. Velocity v vs. ρ obtained from gravity (solid lines) and by solving fluid/gravity viscous hydrodynamics (dashed lines), for $\beta = 0.2, 0.3, 0.5, 0.7$ (*top, black to light grey*) and $\beta = 1, 1.5, 2.$ (*bottom, black to light grey*). The plasma flows from left to right, starting in the same initial equilibrium state for the different β. For small β we see agreement with hydrodynamics. For large β we see strong deviations; for $\beta = 2$ the flow is superluminal in part of the quench region $|\beta\rho| \sim O(1)$.

We see that for the smallest $\beta = 0.2$ the agreement of the gravity stress tensor with that of viscous hydrodynamics is good. The agreement becomes worse as β increases and higher derivative terms in the hydrodynamic expansion become important. For $\beta \simeq O(1)$ the hydrodynamic approximation breaks down and we are in the quench regime. The bulk solutions remains perfectly smooth and allow us to compute the behaviour of this strongly coupled plasma flow. The deviation from hydrodynamics becomes large; for $\beta = 2$ we find the plasma becomes superluminal in a region where the metric is curved, so $2\langle T^{t\rho}\rangle > \langle T^{tt} + T^{\rho\rho}\rangle$ and hence the plasma has no rest frame. The stress tensor vev violates the weak and dominant energy conditions in this region, although all the stress tensor vev components are well behaved - for example in the Supplemental Material we display $\langle T^{tt}\rangle$.

Interestingly we find the equilibrated outgoing plasma has a temperature, and hence entropy density, that is roughly independent of β. The same is true for the fluid/gravity viscous hydrodynamics. One can see in figure 1 that the outgoing velocities v are numerically close for the different β, although they are not obliged to be by stress energy conservation. We emphasise that whilst the total entropy generated in these flows is similar for different β, the region where the spacetime is curved and hence this entropy is generated is very different, becoming small for large β. Hence for strong quenches the entropy density in the plasma is generated in a sudden non-adiabatic manner.

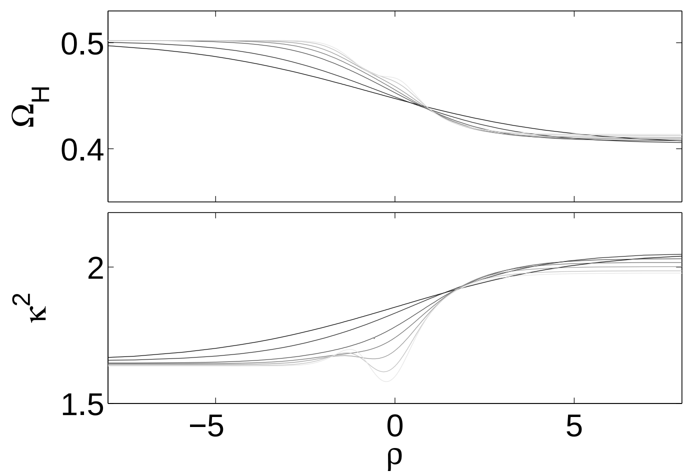

FIG. 2. Velocity of the horizon Ω_H (*top*) and surface gravity κ^2 (*bottom*) as functions of ρ for the same flows as in Fig. 1. These functions explicitly depend on ρ as the horizon is non-Killing. For $\rho \to \pm\infty$ both Ω_H and κ become constant since our solutions approach homogeneous boosted black branes.

The horizon, defined by the zero set of $h(z, \rho) = z - H(\rho)$ is a null surface so that, $g^{\mu\nu}\, \partial_\mu h\, \partial_\nu h = 0$. This is an o.d.e. for $H(\rho)$ which can be solved to find the horizon location. The null tangent to the horizon can be written, $\chi = \frac{\partial}{\partial t} + \Omega_H(\rho) R$, where R has unit norm $R^2 = 1$, is tangent to the horizon and orthogonal to $\partial/\partial t$ and $\partial/\partial y$. Then $\Omega_H(\rho)$ gives the local velocity of the horizon, and is plotted in figure 2. We note this is well behaved even for the flow with $\beta = 2$ which has superluminal boundary stress tensor. The boundary metric, and consequently the bulk metric, explicitly depend on ρ and so $\partial/\partial\rho$ is not Killing. Thus the spacetime motion is not rigid, and hence the local velocity Ω_H explicitly depends on ρ, rather than being constant. We also compute the surface gravity κ defined as $\nabla^\mu(\chi_\nu \chi^\nu) = -2\kappa\chi^\mu$. Again this is not constant, and is plotted in the same figure. It is also well behaved for $\beta = 2$. Further details of the solutions are given in the Supplemental Material.

Discussion.— Perhaps our most striking result is that for a strong quench with $\beta = 2$ we find localised superluminal plasma flow and associated violation of energy conditions. We presume this indicates these flows become dynamically unstable for sufficient quench strength. It is possible this instability is turbulent in nature, in analogy with global AdS-Kerr which may also have superluminal dual plasma [44, 45] with a corresponding superradiant instability that is conjectured to be turbulent [19, 46, 47].

Note added.— [48] appeared simultaneously with this work, finding flowing funnels with non-Killing horizons.

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