Imperial College London

MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Monday 20th May 2013: 14:00 to 16:00

The paper consists of two *sections: A and B*
Section A contains one *question [40 marks total].*
Section B contains four *questions [30 marks each].*

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO QUESTIONS** *from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_{\alpha} g_{\nu\beta} + \partial_{\beta} g_{\alpha\nu} - \partial_{\nu} g_{\alpha\beta} \right)$$

The covariant derivative of a vector field is,

$$\nabla_{\mu} v^{\nu} \equiv \partial_{\mu} v^{\nu} + \Gamma^{\nu}{}_{\mu\alpha} v^{\alpha}$$

and for a covector field is,

$$\nabla_{\mu} w_{\nu} \equiv \partial_{\mu} w_{\nu} - \Gamma^{\alpha}{}_{\mu\nu} w_{\alpha}$$

For a Lagrangian of a curve $x^{\mu}(\lambda)$ of the form,

$$L = \int d\lambda \, \mathcal{L}(x^{\mu}, \frac{dx^{\mu}}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^{\mu}}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^{\mu}}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns the covariant derivative.

(i) State how the components of a $(1,0)$ tensor v^μ and a $(0,1)$ tensor w_μ transform under a coordinate transformation $x \to x'$.

ANSWER:

$$v'^{\mu'} = M^{\mu'}{}_{\mu} v^\mu \ , \quad w'_{\mu'} = M^{\mu}{}_{\mu'} w_\mu$$

where,

$$M^{\mu'}{}_{\mu} = \frac{\partial x'^{\mu'}}{\partial x^\mu} \ , \quad M^{\mu}{}_{\mu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}}$$

[4 marks]

(ii) Use your previous answer to show that $v^\mu w_\mu$ transforms as a scalar under a coordinate transformation $x \to x'$.

ANSWER:

$$v'^{\mu'} w'_{\mu'} = v^\mu M^{\mu'}{}_{\mu} M^{\nu}{}_{\mu'} w_\nu = v^\mu w_\mu$$

as,

$$M^{\mu'}{}_{\mu} M^{\nu}{}_{\mu'} = \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x^\nu}{\partial x'^{\mu'}} = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu$$

[4 marks]

(iii) Under a coordinate transformation the Christoffel symbol transforms as;

$$\Gamma'^{\mu'}{}_{\alpha'\beta'} = \Gamma^{\mu}{}_{\alpha\beta} \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}} - \left(\frac{\partial^2 x'^{\mu'}}{\partial x^\alpha \partial x^\beta} \right) \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}}$$

Does the Christoffel symbol transform as a tensor?

ANSWER: No. If it were a tensor it would transform as;

$$\Gamma'^{\mu'}{}_{\alpha'\beta'} = \Gamma^{\mu}{}_{\alpha\beta} \frac{\partial x'^{\mu'}}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x'^{\alpha'}} \frac{\partial x^\beta}{\partial x'^{\beta'}}$$

ie. missing the second term above. This remaining term is not that of a tensor transformation

[4 marks]

[This question continues on the next page . . .]

(iv) Show that $\partial_\mu w_\nu$, the partial derivative of a covector field w_μ, does *not* transform as a tensor.

ANSWER:

$$\partial_{\mu'} w'_{\nu'} = \frac{\partial}{\partial x'^{\mu'}}\left(\frac{\partial x^\nu}{\partial x'^{\nu'}} w_\nu\right)$$

Using chain rule,

$$\frac{\partial}{\partial x'^{\mu'}} = \frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial}{\partial x^\mu}$$

then,

$$\begin{aligned}\partial_{\mu'} w'_{\nu'} &= \frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x'^{\nu'}} w_\nu\right) \\ &= \frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial x^\nu}{\partial x'^{\nu'}}\partial_\mu w_\nu + w_\nu \frac{\partial x^\mu}{\partial x'^{\mu'}}\frac{\partial}{\partial x^\mu}\left(\frac{\partial x^\nu}{\partial x'^{\nu'}}\right)\end{aligned}$$

The first term is the usual tensor transformation for a $(0,2)$ tensor. However, in addition to this, there is also the second term which is not part of the usual tensor transformation.

[4 marks]

(v) Starting from the identity,

$$\delta^\mu_\nu = \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^\nu} \tag{1}$$

take an appropriate partial derivative of this to derive,

$$\frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial x'^\alpha}{\partial x^\alpha}\frac{\partial x^\mu}{\partial x'^\alpha \partial x'^{\nu'}} = -\frac{\partial x^\mu}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^\alpha \partial x^\nu} \tag{2}$$

ANSWER: Taking a derivative ∂_α;

$$\partial_\alpha \delta^\mu_\nu = \partial_\alpha\left(\frac{\partial x^\mu}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^\nu}\right) = \frac{\partial x'^{\nu'}}{\partial x^\nu}\partial_\alpha \frac{\partial x^\mu}{\partial x'^{\nu'}} + \frac{\partial x^\mu}{\partial x'^{\nu'}}\partial_\alpha \frac{\partial x'^{\nu'}}{\partial x^\nu} \tag{3}$$

Now $\partial_\alpha \delta^\mu_\nu = 0$ and so,

$$\begin{aligned} 0 &= \frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial}{\partial x^\alpha}\frac{\partial x^\mu}{\partial x'^{\nu'}} + \frac{\partial x^\mu}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^\alpha \partial x^\nu} \\ &= \frac{\partial x'^{\nu'}}{\partial x^\nu}\frac{\partial x'^\alpha}{\partial x^\alpha}\frac{\partial}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^{\nu'}} + \frac{\partial x^\mu}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^\alpha \partial x^\nu} \end{aligned} \tag{4}$$

[This question continues on the next page . . .]

So that,

$$\frac{\partial x'^{\nu'}}{\partial x^{\nu}}\frac{\partial x'^{\alpha}}{\partial x^{\alpha}}\frac{\partial x^{\mu}}{\partial x'^{\alpha}\partial x'^{\nu'}} = -\frac{\partial x^{\mu}}{\partial x'^{\nu'}}\frac{\partial x'^{\nu'}}{\partial x^{\alpha}\partial x^{\nu}} \tag{5}$$

[4 marks]

(vi) Show that the covariant derivative of a covector field w_μ, defined as $\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$, does transform as a tensor.

ANSWER: From the previous parts;

$$\begin{aligned}
\nabla_{\mu'} w'_{\nu'} &= \partial_{\mu'} w'_{\nu'} - \Gamma'^{\alpha'}{}_{\mu'\nu'} w'_{\alpha'} \\
&= \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\partial_\mu w_\nu + w_\nu \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\right) \\
&\quad - \left(w_\alpha \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\right)\left(\Gamma^{\beta}{}_{\mu\nu}\frac{\partial x'^{\alpha'}}{\partial x^{\beta}}\frac{\partial x^{\mu}}{\partial x'^{\nu'}}\frac{\partial x^{\nu}}{\partial x'^{\mu'}}\right) \\
&\quad + \left(w_\alpha \frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\right)\left(\left(\frac{\partial^2 x'^{\alpha'}}{\partial x^{\rho}\partial x^{\sigma}}\right)\frac{\partial x^{\rho}}{\partial x'^{\mu'}}\frac{\partial x^{\sigma}}{\partial x'^{\nu'}}\right) \\
&= \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\left(\partial_\mu w_\nu - w_\alpha \Gamma^{\beta}{}_{\mu\nu}\frac{\partial x'^{\alpha'}}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\right) \\
&\quad + w_\alpha\left(\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu'}\partial x'^{\nu'}} + \left(\frac{\partial^2 x'^{\alpha'}}{\partial x^{\rho}\partial x^{\sigma}}\right)\frac{\partial x^{\rho}}{\partial x'^{\mu'}}\frac{\partial x^{\sigma}}{\partial x'^{\nu'}}\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\right)
\end{aligned}$$

From previous part;

$$\frac{\partial x'^{\alpha'}}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu'}\partial x'^{\nu'}} = -\left(\frac{\partial^2 x'^{\alpha'}}{\partial x^{\rho}\partial x^{\sigma}}\right)\frac{\partial x^{\rho}}{\partial x'^{\mu'}}\frac{\partial x^{\sigma}}{\partial x'^{\nu'}}$$

and hence

$$\frac{\partial^2 x^{\alpha}}{\partial x'^{\mu'}\partial x'^{\nu'}} = -\left(\frac{\partial^2 x'^{\alpha'}}{\partial x^{\rho}\partial x^{\sigma}}\right)\frac{\partial x^{\rho}}{\partial x'^{\mu'}}\frac{\partial x^{\sigma}}{\partial x'^{\nu'}}\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}$$

Using this we have;

$$\begin{aligned}
\nabla_{\mu'} w'_{\nu'} &= \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\left(\partial_\mu w_\nu - w_\alpha \Gamma^{\alpha}{}_{\mu\nu}\right) \\
&= \frac{\partial x^{\mu}}{\partial x'^{\mu'}}\frac{\partial x^{\nu}}{\partial x'^{\nu'}}\nabla_\mu w_\nu
\end{aligned}$$

as required for a tensor.

[4 marks]

[Total 24 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Newtonian spacetime, which we write using coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ as,

$$ds^2 = \left(\eta_{\mu\nu} - 2\epsilon\Phi(x)\delta_{\mu\nu}\right) dx^\mu dx^\nu$$

where $\epsilon\Phi$ is the Newtonian potential, and we are interested in the Newtonian limit $\epsilon \to 0$ so that $|\epsilon\Phi| \ll 1$.

(i) State the stress tensor for a perfect fluid in a general spacetime in terms of its energy density ρ, pressure P and local 4-velocity u^μ (where $u^\mu u_\mu = -1$).

ANSWER:

$$T_{\mu\nu} = (\rho + P)\, u_\mu u_\nu + P\, g_{\mu\nu}$$

[1 mark]

(ii) In the limit $\epsilon \to 0$ the components of the Ricci tensor to leading order in ϵ are;

$$\begin{aligned} R_{tt} &= \epsilon\, \delta_{ij}\partial_i\partial_j\Phi \\ R_{ti} &= 0 \\ R_{ij} &= \epsilon\, \delta_{ij}\ (\delta_{ab}\partial_a\partial_b\Phi) \end{aligned}$$

Use these to compute the components of the stress tensor that satisfies the Einstein equations for this spacetime. Show that this is the stress tensor for a dust fluid (ie. fluid with zero pressure), and determine the 4-velocity and energy density of this dust in terms of the Newtonian potential $\epsilon\Phi$.

ANSWER: Then $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. The trace,

$$R = g^{\mu\nu} R_{\mu\nu} = g^{tt} R_{tt} + 2 g^{ti} R_{ti} + g^{ij} R_{ij}$$

Since $R_{\mu\nu}$ is already $O(\epsilon)$, then to leading order $O(\epsilon)$ then,

$$\begin{aligned} R &= \eta^{tt} R_{tt} + \eta^{ij} R_{ij} \\ &= -R_{tt} + \delta_{ij} R_{ij} \\ &= -\left(\epsilon\, \delta_{ij}\partial_i\partial_j\Phi\right) + \delta_{ij}\left(\epsilon\, \delta_{ij}\ (\delta_{ab}\partial_a\partial_b\Phi)\right) \\ &= \epsilon\left(-\delta_{ij}\partial_i\partial_j\Phi + \delta_{ij}\delta_{ij}\,(\delta_{ab}\partial_a\partial_b\Phi)\right) \end{aligned}$$

Now recall that $\delta_{ij}\delta_{ij} = 3$, then,

$$\begin{aligned} R &= \epsilon\left(-\delta_{ij}\partial_i\partial_j\Phi + 3\,(\delta_{ab}\partial_a\partial_b\Phi)\right) \\ &= \epsilon\,(2\delta_{ab}\partial_a\partial_b\Phi) \end{aligned}$$

[This question continues on the next page . . .]

Then,

$$\begin{aligned} G_{tt} &= R_{tt} - \frac{1}{2} g_{tt} R \\ &= R_{tt} + \frac{1}{2} R \\ &= \epsilon\, \delta_{ij} \partial_i \partial_j \Phi + \frac{1}{2} \epsilon \left(2\delta_{ab} \partial_a \partial_b \Phi \right) \\ &= \epsilon \left(2\delta_{ab} \partial_a \partial_b \Phi \right) \end{aligned}$$

to leading order.
The off diagonal terms vansh; $G_{ti} = R_{ti} - \frac{1}{2} g_{ti} R = 0$
The spatial components;

$$\begin{aligned} G_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R \\ &= R_{ij} - \frac{1}{2} \delta_{ij} R \\ &= \epsilon\, \delta_{ij} \left(\delta_{ab} \partial_a \partial_b \Phi \right) - \frac{1}{2} \delta_{ij} \epsilon \left(2\delta_{ab} \partial_a \partial_b \Phi \right) \\ &= 0 \end{aligned}$$

also vanish to leading order $O(\epsilon)$.
The Einstein equations ($c = 1$) are,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{1}$$

so then the stress tensor that must satisfy the Einstein equation is;

$$\begin{aligned} T_{tt} &= \frac{1}{8\pi G_N} G_{tt} = \frac{1}{8\pi G_N} \epsilon \left(2\delta_{ab} \partial_a \partial_b \Phi \right) \\ &= \frac{1}{4\pi G_N} \delta_{ij} \partial_i \partial_j \left(\epsilon \Phi \right) \end{aligned}$$

with $T_{ti} = T_{ij} = 0$ to leading order.
A dust fluid has $P = 0$ and so, $T_{\mu\nu} = \rho u_\mu u_\nu$. Taking the fluid to be static (to leading order) so that $u^\mu = (1, 0, 0, 0)$ and hence, $u_\mu = (-1, 0, 0, 0)$ to leading order, then,

$$T_{tt} = \rho u_t u_t = \rho$$

to leading order, and $T_{ti} = T_{ij} = 0$.
Thus equating these, we find;

$$\rho = \frac{1}{4\pi G_N} \delta_{ij} \partial_i \partial_j \left(\epsilon \Phi \right)$$

[This question continues on the next page . . .]

and hence recover the Newton law for gravity,

$$\delta_{ij}\partial_i\partial_j\left(\epsilon\Phi\right) = 4\pi G_N\rho$$

for Newtonian potential $\epsilon\Phi$.

[1 mark]

(iii) By calculation, show that to leading order in ϵ,

$$\Gamma^i{}_{tt} = +\epsilon\partial_i\Phi$$

Using this, show that a non-accelerated particle that is *slowly moving* obeys (to leading order in $\epsilon \to 0$),

$$\frac{d^2x^i}{dt^2} = -\partial_i\left(\epsilon\Phi\right)$$

ANSWER: So,

$$\begin{aligned}
\Gamma^i{}_{tt} &= \frac{1}{2}g^{i\mu}\left(\partial_t g_{\mu t} + \partial_t g_{t\mu} - \partial_\mu g_{tt}\right) \\
&= -\frac{1}{2}g^{ij}\partial_j g_{tt} \\
&= -\frac{1}{2}g^{ij}\partial_j g_{tt}
\end{aligned}$$

Now the inverse metric is $g^{\mu\nu} = (\eta^{\mu\nu} + 2\epsilon\Phi\delta^{\mu\nu})$ to leading order.
Then,

$$\begin{aligned}
\Gamma^i{}_{tt} &= -\frac{1}{2}\left(\eta^{ij} + 2\epsilon\Phi\delta^{ij}\right)\partial_j\left(\eta_{tt} - 2\epsilon\Phi\delta_{tt}\right) \\
&= \epsilon\frac{1}{2}\left(\delta^{ij} + 2\epsilon\Phi\delta^{ij}\right)2\partial_j\Phi \\
&= \epsilon\delta^{ij}\partial_j\Phi \\
&= \epsilon\partial_i\Phi
\end{aligned}$$

Consider geodesic equation;

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

and so taking the spatial component;

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Now for slow motion we consider $\frac{dx^t}{d\tau} \simeq 1$ and $\frac{dx^i}{d\tau} \simeq 0$ to leading order. Then,

$$\frac{d^2x^i}{d\tau^2} + \Gamma^i{}_{tt} = 0$$

[This question continues on the next page . . .]

and hence,

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{tt} = -\partial_i\left(\epsilon\Phi\right)$$

[1 mark]

[Total 3 marks]

3. This question concerns the Schwarschild metric, which we write using coordinates $x^\mu = (t, r, \theta, \phi)$ as,

$$ds^2 = -\left(1 - \frac{2G\,M}{r}\right)dt^2 + \left(1 - \frac{2G\,M}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

for a mass M, with G the Newton constant.

(i) Consider a timelike geodesic $x^\mu(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ in the Schwarzschild metric where τ is proper time. Write a Lagrangian that we may vary to determine the geodesic. Deduce the Euler-Lagrange equations for Θ and Φ. Show these are consistent with a geodesic that lies in the plane $\theta = \pi/2$. We now restrict our attention to such geodesics. Show then that,

$$R^2\frac{d\Phi}{d\tau} = J$$

where J is a constant.

ANSWER:

$$L = \int d\tau \mathcal{L}$$

where

$$\begin{aligned}\mathcal{L} &= g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \\ &= -\left(1 - \frac{2GM}{R}\right)\dot{T}^2 + \left(1 - \frac{2GM}{R}\right)^{-1}\dot{R}^2 + R^2\left(\dot{\Theta}^2 + \sin^2\Theta\,\dot{\Phi}^2\right)\end{aligned}$$

with $\dot{} = d/d\tau$.

Euler-Lagrange (E-L) equation for Θ:

$$\frac{d}{d\tau}\left(2R^2\dot{\Theta}\right) = 2R^2\sin\Theta\cos\Theta\dot{\Phi}^2$$

E-L equation for Φ:

$$\frac{d}{d\tau}\left(2R^2\sin^2\Theta\dot{\Phi}\right) = 0$$

Taking $\Theta = \pi/2$ then the first of these above is satisfied as $\dot{\Theta} = 0$ and $\cos\Theta = 0$. The second becomes;

$$\frac{d}{d\tau}\left(2R^2\dot{\Phi}\right) = 0$$

Hence, $R^2\dot{\Phi}$ =constant.

[1 mark]

[This question continues on the next page . . .]

(ii) Further deduce the equations that govern T and R. Show that,

$$\left(1 - \frac{2GM}{r}\right)\frac{dT}{d\tau} \quad = \quad k$$

were k is a constant. Hence show the equation governing the radial motion in the plane $\theta = \pi/2$ looks like that of one dimensional motion for a unit mass particle in a potential $V(R)$ with constant energy E so,

$$E = \frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V(R)\ , \qquad V(R) = -\frac{GM}{R} + \frac{J^2}{2R^2} + \frac{\alpha J^2}{R^3}$$

where α is a constant depending on the mass M and Newton constant G that you should determine.

ANSWER:

E-L equation for T:

$$\frac{d}{d\tau}\left(-2\left(1 - \frac{2GM}{r}\right)\frac{dT}{d\tau}\right) \quad = \quad 0$$

Hence,

$$\left(1 - \frac{2GM}{r}\right)\frac{dT}{d\tau} \quad = \quad k$$

for constant of integration k.

The remaining equation is best derived from condition $\mathcal{L} = -1$ since the parameter τ is proper time. Then (recalling $\Phi = pi/2$),

$$\begin{aligned} -1 \quad &= \quad -\left(1 - \frac{2GM}{R}\right)\dot{T}^2 + \left(1 - \frac{2GM}{R}\right)^{-1}\dot{R}^2 + R^2\dot{\Phi}^2 \\ &= \quad -\frac{k^2}{\left(1 - \frac{2GM}{R}\right)} + \frac{1}{1 - \frac{2GM}{R}}\dot{R}^2 + \frac{R^2}{J^2} \end{aligned}$$

So,

$$0 \quad = \quad 1 - \frac{k^2}{\left(1 - \frac{2GM}{R}\right)} + \frac{1}{1 - \frac{2GM}{R}}\dot{R}^2 + \frac{J^2}{R^2}$$

then,

$$\begin{aligned} \frac{1}{2}k^2 \quad &= \quad \frac{1}{2}\dot{R}^2 + \frac{1}{2}\left(1 - \frac{2GM}{R}\right)\left(1 + \frac{J^2}{R^2}\right) \\ &= \quad \frac{1}{2}\dot{R}^2 + \frac{1}{2} - \frac{GM}{R} + \frac{J^2}{2R^2} - \frac{GMJ^2}{R^3} \end{aligned}$$

[This question continues on the next page . . .]

and so,

$$E = \frac{1}{2}k^2 - \frac{1}{2} = \frac{1}{2}\dot{R}^2 - \frac{GM}{R} + \frac{J^2}{2R^2} - \frac{GMJ^2}{R^3}$$

So,

$$V(R) = -\frac{GM}{R} + \frac{J^2}{2R^2} - \frac{GMJ^2}{R^3}$$

so $\alpha = -GM$. For Newtonian gravity $\alpha = 0$.

[1 mark]

(iii) Show that for a circular orbit, with constant radius $R = R_0$, then,

$$V''(R_0) = \frac{J^2}{R_0^4}\left(1 + \frac{6\alpha}{R_0}\right) \quad (1)$$

ANSWER:

For a unit mass particle in a potential $V(R)$,

$$\ddot{R} = -V'(R) \quad (2)$$

and for a circular orbit R =constant, so $\ddot{R} = 0$ so $V'(R) = 0$.

So,

$$V(R) = -\frac{GM}{R} + \frac{J^2}{2R^2} + \frac{\alpha J^2}{R^3}$$

then,

$$V'(R) = +\frac{GM}{R^2} - \frac{J^2}{R^3} - \frac{3\alpha J^2}{R^4}$$

and,

$$V''(R) = -\frac{2GM}{R^3} + \frac{3J^2}{R^4} + \frac{12\alpha J^2}{R^5}$$

For a circular orbit $R = R_0$ then,

$$\frac{J^2}{R_0}\left(1 + \frac{3\alpha}{R_0}\right) = GM$$

so that,

$$\begin{aligned} V''(R_0) &= -\frac{2GM}{R_0^3} + J^2\left(\frac{3}{R_0^4} + \frac{12\alpha}{R_0^5}\right) \\ &= -\frac{2J^2}{R_0^4}\left(1 + \frac{3\alpha}{R_0}\right) + J^2\left(\frac{3}{R_0^4} + \frac{12\alpha}{R_0^5}\right) \\ &= \frac{J^2}{R_0^4}\left(-2\left(1 + \frac{3\alpha}{R_0}\right) + 3 + \frac{12\alpha}{R_0}\right) \\ &= \frac{J^2}{R_0^4}\left(1 + \frac{6\alpha}{R_0}\right) \end{aligned}$$

[1 mark]

[This question continues on the next page . . .]

(iv) Compute the proper time T_{ang} required for Φ to traverse an angle 2π. Show that for a circular orbit radius $R = R_0$ that is perturbed a little, so $R(\tau) \simeq R_0 + \delta R(\tau)$, the motion approximately performs simple harmonic oscillation with period,

$$T_{rad} = \frac{2\pi}{\sqrt{V''(R_0)}}$$

Comment on the relation between T_{ang} and T_{rad}.

ANSWER:

From,

$$R^2 \frac{d\Phi}{d\tau} = J$$

the proper time for a circular orbit, T_{ang}, is;

$$T_{ang} = \frac{2\pi R_0^2}{J}$$

as $R = R_0$ =constant.

For a unit mass particle in a potential $V(R)$,

$$\ddot{R} = -V'(R)$$

If $R(\tau) \simeq R_0 + \delta R(\tau)$ for R_0 a circular orbit $V'(R_0) = 0$, then we can expand,

$$\begin{aligned} V'(R) &= V'(R_0 + \delta R(\tau)) = V'(R_0) + \delta R(\tau) V''(R_0) + ... \\ &= \delta R(\tau) V''(R_0) \end{aligned}$$

so that,

$$\ddot{R} = \delta \ddot{R}(\tau) \simeq -\delta R(\tau) V''(R_0)$$

This is SHO with period,

$$T_{rad} = \frac{2\pi}{\sqrt{V''(R)}}$$

so,

$$T_{rad} = \frac{2\pi R^2}{J} \frac{1}{\sqrt{1 + \frac{6\alpha}{R}}}$$

$T_{rad} = T_{ang}$ for Newton theory $\alpha = 0$, and hence have closed orbits when perturbed from circularity. However for GR they are not the same, so the orbit does not close, hence the perihelion precesses.

[1 mark]

[Total 4 marks]

4. (i) Consider a particle following a timelike curve $x^\mu(\tau)$ in a general spacetime, where τ is the particle's proper time. The 4-velocity $v^\mu = dx^\mu/d\tau$. Give the expression for the 4-acceleration a^μ in terms of v^μ and its covariant derivative.

ANSWER:

$$a^\mu = v^\nu \nabla_\nu v^\mu$$

[1 mark]

(ii) Show that for the case of Minkowski spacetime in Minkowski coordinates $x^\mu = (t, x^i)$ so that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ then this reduces to the Special Relativity result,

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} \tag{1}$$

ANSWER:

In Minkowski spacetime in canonical coordinates so that $g_{\mu\nu} = \eta_{\mu\nu}$ then $\Gamma^\mu{}_{\alpha\beta} = 0$. Then,

$$\begin{aligned} a^\mu &= v^\nu \nabla_\nu v^\mu = v^\nu \partial_\nu v^\mu + v^\nu v^\alpha \Gamma^\mu{}_{\nu\alpha} = v^\nu \partial_\nu v^\mu \\ &= \frac{dx^\nu}{d\tau} \frac{\partial}{\partial x^\nu} v^\mu = \frac{d}{d\tau} v^\mu = \frac{d^2}{d\tau^2} x^\mu \end{aligned} \tag{2}$$

[1 mark]

(iii) By carefully varying the action,

$$L = \int d\tau \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \tag{3}$$

show that the Euler-Lagrange equations are related to the geodesic condition $v^\mu \nabla_\mu v^\nu = 0$ as,

$$2 v^\mu \nabla_\mu v_\alpha = \frac{d}{d\tau} \left(\frac{\partial \mathcal{L}}{\partial \frac{dx^\alpha}{d\tau}} \right) - \frac{\partial \mathcal{L}}{\partial x^\alpha} \tag{4}$$

ANSWER:

The geodesic condition,

$$\begin{aligned} v^\mu \nabla_\mu v^\nu &= \frac{dx^\mu}{d\tau} \left(\partial_\mu v^\nu + \Gamma^\nu{}_{\mu\alpha} v^\alpha \right) \\ &= \frac{dx^\mu}{d\tau} \frac{\partial}{\partial x^\mu} v^\nu + \Gamma^\nu{}_{\mu\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\alpha}{d\tau} \\ &= \frac{dv^\nu}{d\tau} + \Gamma^\nu{}_{\mu\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\alpha}{d\tau} \\ &= \frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu{}_{\mu\alpha} \frac{dx^\mu}{d\tau} \frac{dx^\alpha}{d\tau} \end{aligned} \tag{5}$$

[This question continues on the next page . . .]

Now,

$$L = \int d\tau \mathcal{L} , \qquad \mathcal{L} = g_{\mu\nu}(x) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$

Firstly;

$$\frac{\partial \mathcal{L}}{\partial x^\alpha} = \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\alpha} g_{\mu\nu}(x)$$

and secondly,

$$\frac{\partial \mathcal{L}}{\partial \frac{dx^\alpha}{d\tau}} = 2\frac{dx^\nu}{d\tau} g_{\alpha\nu}(x)$$

Then,

$$\begin{aligned}
\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \frac{dx^\alpha}{d\tau}} &= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu}(x) + 2\frac{dx^\nu}{d\tau}\frac{d}{d\tau} g_{\alpha\nu}(x) \\
&= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu}(x) + 2\frac{dx^\nu}{d\tau}\frac{dx^\beta}{d\tau}\frac{d}{dx^\beta} g_{\alpha\nu}(x)
\end{aligned}$$

Then;

$$\begin{aligned}
\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}}{\partial x^\alpha} &= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu}(x) + 2\frac{dx^\nu}{d\tau}\frac{dx^\beta}{d\tau}\frac{d}{dx^\beta} g_{\alpha\nu}(x) - \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\alpha} g_{\mu\nu}(x) \\
&= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu} + 2\frac{dx^\nu}{d\tau}\frac{dx^\beta}{d\tau}\frac{d}{dx^\beta} g_{\alpha\nu} - \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial}{\partial x^\alpha} g_{\mu\nu} \\
&= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu} + \left(2\frac{d}{dx^\mu} g_{\alpha\nu} - \frac{\partial}{\partial x^\alpha} g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
&= 2\frac{d^2x^\nu}{d\tau^2} g_{\alpha\nu} + \left(\frac{d}{dx^\mu} g_{\alpha\nu} + \frac{d}{dx^\nu} g_{\alpha\mu} - \frac{\partial}{\partial x^\alpha} g_{\mu\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
&= 2g_{\alpha\nu}\frac{d^2x^\nu}{d\tau^2} + 2g_{\alpha\beta}\Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \\
&= 2g_{\alpha\beta}\left(\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right)
\end{aligned}$$

So comparing equations (5) and (6) we obtain,

$$\begin{aligned}
2v^\mu \nabla_\mu v_\alpha &= 2g_{\alpha\beta} v^\mu \nabla_\mu v^\beta = 2g_{\alpha\beta}\left(\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) \\
&= \frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}}{\partial x^\alpha}
\end{aligned}$$

as required.

[1 mark]

[This question continues on the next page . . .]

(iv) Consider now a particle coupled to a vector field $A_\mu(x)$ in a general spacetime so that its Lagrangian is modified to,

$$L = \int d\tau \left(g_{\mu\nu}(x) \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + A_\mu(x)\frac{dx^\mu}{d\tau} \right) \tag{6}$$

Show that the 4-acceleration of the particle is;

$$a^\mu = \frac{1}{2} F^{\mu\nu} v_\nu \ , \qquad F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{7}$$

ANSWER:

Let us split the action up into $\mathcal{L}_{free}$ and $\mathcal{L}_{int}$;

$$L = \int d\tau \mathcal{L}_{free} + \mathcal{L}_{int} \ , \qquad \mathcal{L}_{free} = g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \ , \qquad \mathcal{L}_{int} = A_\mu(x)\frac{dx^\mu}{d\tau} \tag{8}$$

The E-L equations are now;

$$\begin{aligned} 0 &= \left(\frac{d}{d\tau}\frac{\partial \mathcal{L}_{free}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}_{free}}{\partial x^\alpha} \right) + \left(\frac{d}{d\tau}\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}_{int}}{\partial x^\alpha} \right) \\ &= 2v^\mu \nabla_\mu v_\alpha + \left(\frac{d}{d\tau}\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}_{int}}{\partial x^\alpha} \right) \\ &= 2a_\alpha + \left(\frac{d}{d\tau}\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}_{int}}{\partial x^\alpha} \right) \end{aligned}$$

Hence we obtain the acceleration from the variation;

$$a_\alpha = -\frac{1}{2}\left(\frac{d}{d\tau}\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} - \frac{\partial \mathcal{L}_{int}}{\partial x^\alpha} \right)$$

For $\mathcal{L}_{int} = A_\mu(x)\frac{dx^\mu}{d\tau}$ we have,

$$\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} = A_\alpha(x) \ , \quad \frac{\partial \mathcal{L}_{int}}{\partial x^\alpha} = \frac{dx^\mu}{d\tau}\partial_\alpha A_\mu(x)$$

and so,

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}_{int}}{\partial \frac{dx^\alpha}{d\tau}} = \frac{d}{d\tau}A_\alpha(x) = \frac{dx^\mu}{d\tau}\partial_\mu A_\alpha(x)$$

Then,

$$\begin{aligned} a_\alpha &= -\frac{1}{2}\left(\frac{dx^\mu}{d\tau}\partial_\mu A_\alpha(x) - \frac{dx^\mu}{d\tau}\partial_\alpha A_\mu(x) \right) \\ &= \frac{1}{2}\frac{dx^\mu}{d\tau}\left(\partial_\alpha A_\mu(x) - \partial_\mu A_\alpha(x) \right) \end{aligned}$$

[This question continues on the next page . . .]

Finally, note that,

$$\begin{aligned} F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu \\ &= \partial_\mu A_\nu - \Gamma^\alpha{}_{\mu\nu} A_\alpha - \partial_\nu A_\mu + \Gamma^\alpha{}_{\nu\mu} A_\alpha \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned}$$

and hence we see,

$$a_\alpha = \frac{1}{2}\frac{dx^\mu}{d\tau}\Big(\partial_\alpha A_\mu(x) - \partial_\mu A_\alpha(x)\Big) = \frac{1}{2} F_{\alpha\mu} v^\mu$$

as required.

[1 mark]

[Total 4 marks]

5. (i) Show that the Christoffel symbol is related to partial derivatives of the metric as,

$$\partial_\alpha g_{\mu\nu} = g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} + g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}$$

ANSWER:
Now,

$$\begin{aligned} g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} &= g_{\mu\beta}\left(\frac{1}{2}g^{\beta\sigma}\left(\partial_\nu g_{\alpha\sigma} + \partial_\alpha g_{\sigma\nu} - \partial_\sigma g_{\alpha\nu}\right)\right) \\ &= \frac{1}{2}\left(\partial_\nu g_{\alpha\mu} + \partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu}\right) \end{aligned}$$

So,

$$\begin{aligned} g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} + g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu} &= \frac{1}{2}\left(\partial_\nu g_{\alpha\mu} + \partial_\alpha g_{\mu\nu} - \partial_\mu g_{\alpha\nu}\right) + \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\alpha g_{\nu\mu} - \partial_\nu g_{\alpha\mu}\right) \\ &= \frac{1}{2}\partial_\alpha g_{\mu\nu} + \frac{1}{2}\partial_\alpha g_{\nu\mu} = \partial_\alpha g_{\mu\nu} \end{aligned}$$

as required.

(ii) The Lie derivative of a $(0, 2)$ tensor $A_{\mu\nu}$ with respect to a vector field w^μ is,

$$(Lie)(w, A)_{\mu\nu} = w^\alpha\partial_\alpha A_{\mu\nu} + A_{\mu\alpha}\partial_\nu w^\alpha + A_{\alpha\nu}\partial_\mu w^\alpha$$

Suppose we consider the Lie derivative of the metric $g_{\mu\nu}$. Show that this can also be written in terms of the covariant derivative as,

$$(Lie)(w, g)_{\mu\nu} = \nabla_\mu w_\nu + \nabla_\nu w_\mu$$

If this vanishes, we say w^μ is a *Killing vector field*.

ANSWER:

$$\begin{aligned} (Lie)(w, g)_{\mu\nu} &= w^\alpha\partial_\alpha g_{\mu\nu} + g_{\mu\alpha}\partial_\nu w^\alpha + g_{\alpha\nu}\partial_\mu w^\alpha \\ &= w^\alpha\left(g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu} + g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}\right) + g_{\mu\alpha}\partial_\nu w^\alpha + g_{\alpha\nu}\partial_\mu w^\alpha \\ &= \left(g_{\mu\alpha}\partial_\nu w^\alpha + w^\alpha g_{\mu\beta}\Gamma^\beta{}_{\alpha\nu}\right) + \left(g_{\alpha\nu}\partial_\mu w^\alpha + w^\alpha g_{\nu\beta}\Gamma^\beta{}_{\alpha\mu}\right) \\ &= g_{\mu\beta}\left(\partial_\nu w^\beta + w^\alpha\Gamma^\beta{}_{\alpha\nu}\right) + g_{\nu\beta}\left(\partial_\mu w^\beta + w^\alpha\Gamma^\beta{}_{\alpha\mu}\right) \\ &= g_{\mu\beta}\nabla_\nu w^\beta + g_{\nu\beta}\nabla_\mu w^\beta \\ &= \nabla_\nu w_\mu + \nabla_\mu w_\nu \end{aligned}$$

[1 mark]

[This question continues on the next page . . .]

(iii) Consider a timelike particle with velocity $v^\mu = dx^\mu/d\tau$ for proper time τ. Suppose it follows a geodesic in a spacetime with a Killing vector field w^μ. Show that the quantity,

$$\phi = -w^\mu v_\mu$$

is constant along the particle's trajectory.

ANSWER:
If constant along the particle's trajectory, then,

$$\begin{aligned} 0 &= \frac{d}{d\tau}\phi = \frac{dx^\alpha}{d\tau}\partial_\alpha \phi = v^\alpha \nabla_\alpha \phi \\ &= -v^\alpha \nabla_\alpha \left(w^\mu v_\mu \right) \\ &= -w^\mu \left(v^\alpha \nabla_\alpha v_\mu \right) - v^\alpha v^\mu \nabla_\alpha w_\mu \end{aligned}$$

The first term vanishes by geodesic condition $v^\alpha \nabla_\alpha v_\mu = 0$, the second since w^μ is Killing, so,

$$0 = v^\mu v^\nu Lie(w, g)_{\mu\nu} = v^\mu v^\nu \left(\nabla_\mu w_\nu + \nabla_\nu w_\mu \right) = 2 v^\mu v^\nu \nabla_\mu w_\nu$$

[1 mark]

(iv) Consider the spacetime with coordinates $x^\mu = (t, x^i)$

$$ds^2 = -N(x) dt^2 + g_{ij}(x) dx^i dx^j \qquad (1)$$

where N and g_{ij} only depend on the spatial coordinates x^i and not time t. Show that there is a Killing vector w^μ for this spacetime and explicitly check that $\text{Lie}(w, g) = 0$. Write down the conserved quantity ϕ for a non-accelerated particle's motion. Is this the energy of the particle as measured by observers sitting at constant spatial position?

ANSWER:

[1 mark]

(v) In the spacetime in equation (1) above write down a Lagrangian that may be varied to deduce geodesic motion in the spacetime. Show using the Euler-Lagrange equations that the quantity ϕ is indeed conserved.

ANSWER:

[1 mark]

[Total 4 marks]