Exam

M.Sc. in Quantum Fields and Fundamental Forces

Differential Geometry

2:00 – 5:00, Monday 30 April, 2012

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Answer **THREE** out of the four questions. Use a separate booklet for each question. Make sure that each booklet carries your name, the course title, and the number of the question attempted.

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Question (1)

(1.i). Consider the Lie group $GL(n, \mathbb{R})$ so that $g \in GL(n, \mathbb{R})$ is an $n \times n$ matrix (with non-zero determinant). We may take the matrix components $\{x^{ij}(g)\}$ as coordinates. Using such coordinates show that the vector field

$$V = v^{ij} x^{ki}(g) \frac{\partial}{\partial x^{kj}(g)}$$

is left invariant, where v^{ij} are components of an $n \times n$ matrix. Use this to give a basis for the left invariant vector fields.

[5 marks]

(1.ii). Use your answer to compute the Lie bracket of two left invariant vector fields, V and W in $GL(n, \mathbb{R})$, giving the components of $[V, W]$ in the above basis.

[4 marks]

(1.iii). Consider a matrix group G embedded into $GL(n, \mathbb{R})$ by an embedding $f : G \to GL(n, \mathbb{R})$. The embedding is chosen so that the left action L obeys $f \cdot L_g = L_{f \cdot g} \cdot f$ for $g \in G$ so that the group structure of G is faithfully embedded into GL. Hence show that for $V \in T_e G$ then $f_\star(X_V) = X_{f_\star V}$.

[4 marks]

(1.iv). Show that under a map $f : \mathcal{M} \to \mathcal{N}$ the push-forward of the Lie bracket of two vector fields A, B on $\mathcal{M}$ is given by the Lie bracket of their push-forwards $f_\star A, f_\star B$, ie. show $f_\star[A, B] = [f_\star A, f_\star B]$.

[4 marks]

(1.v). Use your answers above to show how to compute the structure constants of a matrix group G that may be embedded in $GL(n, \mathbb{R})$ using the Lie bracket of $GL(n, \mathbb{R})$.

[3 marks]

[Total 20 marks]

Question (2)

(2.i). Define *cycle* and *boundary* chains? What is *homology*?

[5 marks]

(2.ii). Give a simplicial complex that triangulates the 2-sphere.
(*Hint:* you may find a complex containing only *four* 2-simplices.)

[3 marks]

(2.iii). Construct the boundary operators ∂_2 and ∂_1 as *matrices* for your triangulation of S^2. Use these to confirm that the boundary operator is nilpotent.

[5 marks]

(2.iv). Explicitly compute the Betti numbers b_0, b_1 and b_2 of S^2 using your triangulation and the matrices ∂_2 and ∂_1 you have constructed. Give a basis for the homology vector spaces.

[7 marks]

[Total 20 marks]

Question (3)

(3.i). Give the definition of a *real* manifold.

[3 marks]

(3.ii). Use Stereographic projection to give an Atlas on the n-sphere, S^n. Check this satisfies the definition of a manifold you gave above.

[5 marks]

(3.iii). Let a Lie group G have a transitive action on a compact manifold $\mathcal{M}$. Take a point $p_0 \in \mathcal{M}$, and its stabilizer subgroup H_{p_0}. Construct a smooth map from the coset manifold G/H_{p_0} to $\mathcal{M}$ which is *invertible*. Be sure to explain *why* it is invertible. What is the relationship between $\mathcal{M}$ and the coset G/H_{p_0}?

[6 marks]

(3.iv). Show that the matrix group $U(n+1)$ has a *transitive* group action on S^{2n+1}. (*Hint:* You may assume the fact that for vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{C}^{n+1}$ such that $\mathbf{v}^\dagger \cdot \mathbf{v} = \mathbf{u}^\dagger \cdot \mathbf{u}$ one may always find a matrix $\mathbf{M} \in U(n+1)$ so that $\mathbf{v} = \mathbf{M} \cdot \mathbf{u}$.)

[3 marks]

(3.v). Use this group action to write the manifold S^{2n+1} as a coset G/H where you should determine G and H.

[3 marks]

[Total 20 marks]

Question (4)

(4.i). Using a coordinate basis define the exterior derivative d of an r-form ω. Show explicitly that d is nilpotent.

[4 marks]

(4.ii). Take $\mathcal{M}$ to be an m-dimensional Riemannian manifold. Using a coordinate basis define the Hodge star, $\star\omega$, of an r-form ω. Show that $\star\star\,\omega = \pm\omega$ where you should determine the sign $\pm$ in terms of m and r.

(*Hint:* $(\det g_{\mu\nu})\epsilon^{\alpha_1\ldots\alpha_r\alpha_{r+1}\ldots\alpha_m}\epsilon_{\alpha_1\ldots\alpha_r\beta_{r+1}\ldots\beta_m} = r!(m-r)!\delta^{[\alpha_{r+1}}_{\beta_{1+r}}\delta^{\alpha_{r+2}}_{\beta_{r+2}}\ldots\delta^{\alpha_m]}_{\beta_m}$)

[5 marks]

(4.iii). Use differential forms to write the equations of electromagnetism (EM) in terms of the field strength 2-form F and current 1-form j. Show these equations imply the current is co-closed - what does this imply physically?

[3 marks]

(4.iv). Given an $(m-2)$-chain b on the manifold $\mathcal{M}$ we may define a quantity $Q \equiv \int_b \star F$ in terms of the EM field strength F. Suppose b is a *boundary* chain and that Q vanishes. Then what condition is placed on the current j?

[3 marks]

(4.v). Given a 2-chain c we may define a quantity $Q' \equiv \int_c F$ in terms of the EM field strength F. Suppose we take $\mathcal{M} = \mathbb{R}^m$, a general field strength F obeying the EM equations, and c to be a *cycle* chain. Then show that Q' always vanishes.

Suppose now we remove the origin point so that $\mathcal{M} = \mathbb{R}^m - \{0\}$. Then for which dimensions m does Q' still vanish for a general F solving the EM equations and for all choices of cycle c?

[5 marks]

[Total 20 marks]

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