Imperial College London

MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

Monday 20th May 2013: 14:00 to 16:00

The paper consists of two *sections: A and B*
Section A contains one *question [40 marks total].*
Section B contains four *questions [30 marks each].*

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO QUESTIONS** *from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative of a vector field is,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

and for a covector field is,

$$\nabla_\mu w_\nu \equiv \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$L = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns the covariant derivative.

(i) State how the components of a $(1,0)$ tensor v^μ and a $(0,1)$ tensor w_μ transform under a coordinate transformation $x \to x'$.

[4 marks]

(ii) Use your previous answer to show that $v^\mu w_\mu$ transforms as a scalar under a coordinate transformation $x \to x'$.

[4 marks]

(iii) Show that $\partial_\mu w_\nu$, the partial derivative of a covector field w_μ, does *not* transform as a tensor.

[4 marks]

(iv) Under a coordinate transformation the Christoffel symbol transforms as;

$$\Gamma'^{\mu'}{}_{\alpha'\beta'} = \Gamma^{\mu}{}_{\alpha\beta}\frac{\partial x'^{\mu'}}{\partial x^{\mu}}\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\frac{\partial x^{\beta}}{\partial x'^{\beta'}} - \left(\frac{\partial^2 x'^{\mu'}}{\partial x^{\alpha}\partial x^{\beta}}\right)\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\frac{\partial x^{\beta}}{\partial x'^{\beta'}}$$

Show that this is not a tensor transformation.

[4 marks]

(v) Use this to show that the covariant derivative of a covector field w_μ, defined as $\nabla_\mu w_\nu = \partial_\mu w_\nu - \Gamma^{\alpha}{}_{\mu\nu} w_\alpha$, does transform as a tensor.

[4 marks]

(vi) Given a vector v^μ then $v^\mu w_\mu$ is a coordinate scalar. Use this fact together with the definition of the covariant derivative in the previous part and also the Liebnitz property (product rule) of covariant derivatives to deduce that the covariant derivative of a vector field is,

$$\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

[4 marks]

[Total 24 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Newtonian spacetime, which we write using coordinates $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ as,

$$ds^2 = \left(\eta_{\mu\nu} - 2\epsilon\Phi(x)\delta_{\mu\nu}\right) dx^\mu dx^\nu$$

where $\epsilon\Phi$ is the Newtonian potential, and we are interested in the Newtonian limit $\epsilon \to 0$ so that $|\epsilon\Phi| \ll 1$.

(i) State the stress tensor for a perfect fluid in a general spacetime in terms of its energy density ρ, pressure P and local 4-velocity.

[1 mark]

(ii) In the limit $\epsilon \to 0$ the components of the Ricci tensor to leading order in ϵ are;

$$\begin{aligned} R_{tt} &= \epsilon\,\delta_{ij}\partial_i\partial_j\Phi \\ R_{ti} &= 0 \\ R_{ij} &= \epsilon\,\delta_{ij}\ (\delta_{ab}\partial_a\partial_b\Phi) \end{aligned}$$

Use these to compute the components of the stress tensor that satisfies the Einstein equations for this spacetime. Show that this is the stress tensor for a dust fluid (ie. fluid with zero pressure), and determine the 4-velocity and energy density of this dust in terms of the Newtonian potential $\epsilon\Phi$.

[1 mark]

(iii) By calculation, show that to leading order in ϵ,

$$\Gamma^i{}_{tt} = +\epsilon\partial_i\Phi$$

Show that a non-accelerated particle that is slowly moving obeys (to leading order in $\epsilon \to 0$),

$$\frac{d^2x^i}{dt^2} = -\partial_i\left(\epsilon\Phi\right)$$

[1 mark]

(iv) Briefly state the physical significance of the results in part ii) and part iii) in understanding how General Relativity is compatible with Newton's law of gravity.

[1 mark]

[Total 4 marks]

3. This question concerns Schwarschild metric, which we write using coordinates $x^\mu = (t, r, \theta, \phi)$ as,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

(i) Consider a timelike geodesic $x^\mu(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ in the Schwarzschild metric where τ is proper time. Write a Lagrangian that we may vary to determine the geodesic.

[1 mark]

(ii) By varying the Lagrangian, deduce the Euler-Lagrange equations for Θ and Φ. Show these are consistent with a geodesic that lies in the plane $\theta = \pi/2$. We now restrict our attention to such geodesics. Show then that,

$$R^2 \frac{d\Phi}{d\tau} = J \tag{1}$$

where J is a constant.

[1 mark]

(iii) Further deduce the equations that govern T and R. Show that,

$$\left(1 - \frac{2M}{r}\right) \frac{dT}{d\tau} = E$$

were E is a constant. Hence show the equation governing the radial motion in the plane $\theta = \pi/2$ looks like that of a one dimensional unit mass particle in a potential $V(R)$ with energy E so,

$$E = \frac{1}{2}\left(\frac{dR}{d\tau}\right)^2 + V(R) \tag{2}$$

where you should determine the potential $V(R)$.

[1 mark]

(iv) Consider a circular orbit, so that R is constant. How is the radius R related to the constants E and J? If the orbit is perturbed a little so that it is no longer circular, show that the orbit does not close to an ellipse as in Newtonian gravity, but instead the perihelion precesses.

[1 mark]

[Total 4 marks]

4. This question concerns particle motion in curved spacetimes.

(i) Consider a particle following a timelike curve $x^\mu(\tau)$ in a general spacetime, where τ is the particle's proper time. The 4-velocity $v^\mu = dx^\mu/d\tau$. Give the expression for the 4-acceleration a^μ in terms of v^μ and its covariant derivative.

[1 mark]

(ii) Show that for the case of Minkowski spacetime in Minkowski coordinates $x^\mu = (t, x^i)$ so that $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ then this reduces to the Special Relativity result,

$$a^\mu = \frac{d^2 x^\mu}{d\tau^2} \tag{1}$$

[1 mark]

(iii) By carefully varying the action,

$$L = \int d\tau \left(g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right) \tag{2}$$

show that the Euler-Lagrange equations derived from it reproduce the geodesic equation,

$$v^\mu \nabla_\mu v^\nu = 0 \tag{3}$$

[1 mark]

(iv) Consider now a particle coupled to a scalar field $\phi(x)$ in a general spacetime, with Lagrangian,

$$L = \int d\tau \left(g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \phi(x) \right) \tag{4}$$

By varying the action, show that it experiences a 4-acceleration,

$$a_\mu = \partial_\mu \phi(x) \tag{5}$$

[1 mark]

(v) Consider now a particle with charge q coupled to an electromagnetic field $A_\mu(x)$ in a general spacetime. Its Lagrangian is modified to,

$$L = \int d\tau \left(g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + q A_\mu(x) \frac{dx^\mu}{d\tau} \right) \tag{6}$$

Compute the 4-acceleration of the particle due to the electromagnetic field.

[1 mark]

[Total 5 marks]

5. This question concerns Killing vectors.

(i) The Lie derivative of a $(0, 2)$ tensor $A_{\mu\nu}$ with respect to a vector field w^μ is defined to be,

$$(Lie)(w, A)_{\mu\nu} = w^\alpha \partial_\alpha A_{\mu\nu} + A_{\mu\alpha}\partial_\nu w^\alpha + A_{\alpha\nu}\partial_\mu w^\alpha$$

Suppose we consider the Lie derivative of the metric $g_{\mu\nu}$. Show that this can also be written in terms of the covariant derivative as,

$$(Lie)(w, g)_{\mu\nu} = \nabla_\mu w_\nu + \nabla_\nu w_\mu$$

If this vanishes, we say w^μ is a Killing vector field.

[1 mark]

(ii) Consider a timelike particle with velocity $v^\mu = dx^\mu/d\tau$ for proper time τ. Suppose it follows a geodesic in a spacetime with a Killing vector field w^μ. Show that the quantity,

$$\phi = -w^\mu v_\mu$$

is constant along the particles trajectory.

[1 mark]

(iii) Consider the spacetime with coordinates $x^\mu = (t, x^i)$

$$ds^2 = -N(x)dt^2 + g_{ij}(x)dx^i dx^j \tag{1}$$

where N and g_{ij} only depend on the spatial coordinates x^i and not time t. Show that there is a Killing vector w^μ for this spacetime and explicitly check that $\mathrm{Lie}(w, g) = 0$. Write down the conserved quantity ϕ for a non-accelerated particle's motion. Is this the energy of the particle as measured by observers sitting at constant spatial position?

[1 mark]

(iv) In the spacetime in equation (1) above write down a Lagrangian that may be varied to deduce geodesic motion in the spacetime. Show using the Euler-Lagrange equations that the quantity ϕ is indeed conserved.

[1 mark]

[Total 4 marks]