

Black hole-black string phase transitions in thermal $1 + 1$ dimensional supersymmetric Yang-Mills theory on a circle

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We review and extend earlier work that uses the AdS/CFT correspondence to relate the black hole-black string transition of gravitational theories on a circle to a phase transition in maximally supersymmetric $1 + 1$ dimensional $SU(N)$ gauge theories at large N , again compactified on a circle. We perform gravity calculations to determine a likely phase diagram for the strongly coupled gauge theory. We then directly study the phase structure of the same gauge theory, now at weak 't Hooft coupling. In the interesting temperature regime for the phase transition, the $1 + 1$ dimensional theory reduces to a $0 + 1$ dimensional bosonic theory, which we solve using Monte Carlo methods. We find strong evidence that the weakly coupled gauge theory also exhibits a black hole-black string like phase transition in the large N limit. We demonstrate that a simple Landau-Ginzburg like model describes the behaviour near the phase transition remarkably well. The weak coupling transition appears to be close to the cusp between a first order and a second order transition.

I. INTRODUCTION

The gravitational theories dual to strongly coupled large N Yang-Mills theories [1, 2, 3] sometimes exhibit particularly dramatic behaviours; they undergo phase transitions involving the nucleation of black holes or transitions between different black holes. Examples include the Hawking-Page transition [4, 5] in AdS spaces and the Gregory-Laflamme transition [6, 7, 8] in the near-horizon geometry of D0-branes on a circle [9, 10, 11, 12], the topic of this letter. These transitions have a dual description as thermal phase transitions in strongly coupled Yang-Mills theories.

It is interesting to investigate how these phase transitions continue to weaker Yang-Mills coupling¹. In the two examples described above, the high temperature phase is (in the large N limit) sharply distinguished from its low temperature counterpart by an order parameter. As a consequence, as we decrease the coupling, the line of phase transitions cannot end abruptly in λ, T space ($\lambda = g_{YM}^2 N$ is the 't Hooft coupling and T is the temperature), and so it must either go to zero or infinite temperature at a finite value of the coupling constant, or (as turns out to be the case in both of the examples described above) it must continue all the way to weak coupling.

The first example described above, the Hawking-Page transition, is dual to a thermal deconfinement transition for the $3 + 1$ dimensional $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on S^3 . It turns out that this transition may indeed be continued to weak coupling [14, 15, 16] where it may be studied in detail with intriguing results.

In this letter we concentrate on the second example, the Gregory-Laflamme transition [6, 7, 8]. Following [9, 10, 11, 12], we first review the dual interpretation of this transition as a thermal phase transition in large N $1 + 1$ -dimensional maximally supersymmetric Yang-Mills (SYM) theory on a circle of circumference L , at strong coupling ($\lambda L^2 \gg 1$)².

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¹ Similar continuations of strong coupling results to weak coupling were recently discussed in [13].

² More precisely, it turns out that the phase transition in this gauge theory is dual to the Gregory-Laflamme transition of a collection of near extremal charged black holes in a space which is asymptotically $R^{8,1} \times S^1$. The transition may be described by a gauge theory,

The uniform black string corresponds to a phase in which the eigenvalues of the holonomy of the gauge field around the spatial circle are uniformly distributed on a unit circle in the complex plane. The black hole corresponds to a phase in which these eigenvalues are clumped about a particular point on the circle. We then demonstrate that Yang-Mills theory on a circle at weak coupling ($\lambda L^2 \ll 1$) undergoes a phase transition distinguished by the same order parameter. We conjecture that this phase transition is the continuation to weak coupling of the black hole - black string transition.

Our analysis of the thermal behaviour of maximally supersymmetric $SU(N)$ Yang-Mills theory on a circle at small λL^2 proceeds in two stages. We first demonstrate that when $L^3 \lambda T \ll 1$ (and also $T^3 L \gg \lambda$), the eigenvalues are sharply localized about a point on the unit circle in the complex plane, in an arc of length $s \sim (L^3 \lambda T)^{1/4}$. Continuing to higher temperatures, it is plausible that at $L^3 \lambda T$ of order one these eigenvalues fill out the circle and undergo a Gross-Witten-like [17, 18] ‘black hole-black string’ transition³. However, perturbation theory breaks down precisely at $T \sim 1/\lambda L^3$. In fact, at high temperatures the two-dimensional theory under study effectively reduces to a one-dimensional Yang-Mills theory with adjoint scalar fields, with ‘t Hooft coupling constant λT , compactified on a circle of circumference L . The effective coupling of this one-dimensional system, $L^3 \lambda T$, climbs to unity precisely when the eigenvalue dynamics gets interesting. Since we do not know how to study this system analytically, the second stage of our analysis involves a Monte Carlo simulation of this effective strongly coupled 0+1-dimensional bosonic system. We find good evidence that there is indeed a sharp phase transition from a localized to a uniform (smeared) eigenvalue distribution, at $T \approx 1.4/\lambda L^3$, and we analyze it in detail. To conclude our note we present a Landau-Ginzburg model that reproduces our detailed numerical results for the eigenvalue dynamics with remarkable accuracy.

The work reported in this letter is part of a more comprehensive investigation of the thermal properties of large N Yang-Mills theories on tori that will appear in a separate publication [19]. In this note we will present the principle results that have bearing on the Gregory-Laflamme transition. Section II contains an analysis of the gravitational system and its translation to Yang-Mills theory. Our main results are in section III, where we analyze the Yang-Mills theory at weak coupling. We end with a summary in section IV. Two appendices contain technical results which are used in section II.

II. HOLOGRAPHIC DUALITY

In this section we review and extend the arguments of [9, 10, 11, 12] that relate the thermodynamics of strongly coupled 1+1-dimensional gauge theories in the ‘t Hooft large N limit to the Gregory-Laflamme transition. In gravity this transition is usually discussed as a function of energy (mass), but in the gauge theory we can also discuss it as a function of temperature (using the relation between the canonical and micro-canonical ensembles), and we will use both languages interchangeably. Consider a 1+1 dimensional maximally supersymmetric Yang-Mills theory at temperature T on a circle of circumference L , in the ‘t Hooft large N limit with ‘t Hooft coupling λ . Define a dimensionless coupling $\lambda' \equiv \lambda L^2$ and a dimensionless temperature $t \equiv TL$. The Maldacena dual of this system is string theory on the space that is obtained upon performing a circle identification $\theta \equiv \theta + 2\pi$ on the familiar background [20] which is the near-horizon limit of an infinitely extended near extremal $D1$ -brane, with the string frame metric and the dilaton given by

$$ds^2 = \alpha' \left\{ \frac{u^3}{\sqrt{d_1 \lambda'}} \left[- \left(1 - \frac{u_0^6}{u^6} \right) \frac{d\tau^2}{L^2} + \frac{d\theta^2}{(2\pi)^2} \right] + \frac{\sqrt{d_1 \lambda'}}{u^3 \left(1 - \frac{u_0^6}{u^6} \right)} du^2 + u^{-1} \sqrt{d_1 \lambda'} d\Omega_7^2 \right\},$$

$$e^\phi = 2\pi \frac{\lambda'}{N} \sqrt{\frac{d_1 \lambda'}{u^6}}, \quad (1)$$

even though the gauge theory is only dual to the near-horizon limit of the black holes, because the unstable mode is localized purely within the near-horizon geometry, as we will show below.

³ In this paper what we mean by a Gross-Witten transition is any transition in which the eigenvalue distribution develops a gap; as discussed in [15] such a transition does not have to be continuous.

where

$$u_0^6 = 2^7 3 \pi^5 \epsilon \left(\frac{\lambda'}{N} \right)^2, \quad d_1 = 2^6 \pi^3, \quad (2)$$

and the dimensionless energy, ϵ , is given by $\epsilon = EL$ (E is the ADM energy of the solution above extremality; note that we are using a dimensionless u coordinate as opposed to the conventions of [20]). There is also a non-trivial RR 3-form field strength. The entropy of the solution, as a function of energy, is given by

$$S = \frac{2^{2/3} \pi^{5/6} N^{2/3} \epsilon^{2/3}}{3^{1/3} \lambda'^{1/6}}. \quad (3)$$

Using (3), one can obtain u_0 as a function of the dimensionless temperature t ,

$$u_0^2 = \frac{16 \pi^{5/2}}{3} t \sqrt{\lambda'}. \quad (4)$$

Under what conditions are the stringy corrections to the (non-supersymmetric) supergravity solution (1) negligible? In the neighbourhood of its horizon, (1) is characterized by a single length scale $l \sim \sqrt{\alpha'} \lambda'^{1/4} u_0^{-1/2}$. Consequently, α' corrections to (1) are negligible when $l \gg \sqrt{\alpha'}$, i.e. when $t \ll \sqrt{\lambda'}$. Winding modes (whose mass in the neighbourhood of the horizon is $M_w \sim u_0^{3/2} / \sqrt{\alpha'} \lambda'^{1/4}$) are negligible when $l M_w \gg 1$, i.e. for $t \gg 1/\sqrt{\lambda'}$. Thus, when $\lambda' \gg 1$, the solution is valid over a large range of temperatures.

In order to obtain a supergravity description that is valid for t lower than or of order $1/\sqrt{\lambda'}$, we T-dualize in the θ direction to obtain the metric for a collection of non-extremal D0-branes on a dual circle. In general, the distribution of these 0-branes around the circle is dynamically determined. At large enough temperatures, they are uniformly distributed over the circle, and the corresponding supergravity background is the T-dual of (1),

$$\begin{aligned} ds^2 &= \alpha' \left(-u^3 \left(1 - \frac{u_0^6}{u^6} \right) \frac{d\tau^2}{L^2 \sqrt{\lambda' d_1}} + \sqrt{d_1 \lambda'} u^{-1} d\Omega_7^2 + \frac{\sqrt{d_1 \lambda'}}{u^3} \left[\frac{du^2}{\left(1 - \frac{u_0^6}{u^6} \right)} + (2\pi)^2 d\tilde{\theta}^2 \right] \right), \\ e^\phi &= (2\pi)^2 \frac{\lambda'}{N} \left(d_1 \frac{\lambda'}{u^6} \right)^{\frac{3}{4}}, \end{aligned} \quad (5)$$

where again $\tilde{\theta} \equiv \tilde{\theta} + 2\pi$ and there is a non-trivial RR 2-form field strength. As above, α' corrections to the supergravity solution (5) (in the neighbourhood of the horizon) are negligible for $t \ll \sqrt{\lambda'}$; winding modes about this background are negligible provided $t \ll 1$. In summary, stringy corrections to the background (5) are negligible at large λ' provided $t \ll 1$.

Below we will also be interested in fluctuation modes about (5) that carry momentum about the circle. Such modes will not excite string oscillators (and so will be well described by supergravity) if and only if the proper length of the compact circle (near the horizon) is large in string units, i.e. for $t \ll 1/\lambda'^{1/6}$.

A. The uniform phase and the Gregory-Laflamme instability

Equation (5) is the near-horizon geometry of a charged black string in $R^{8,1} \times S^1$ (winding around the S^1); in this section, we demonstrate that it develops a Gregory-Laflamme instability at a temperature $t \sim 1/\sqrt{\lambda'}$, which is well within the validity of the supergravity approximation for large λ' . In order to facilitate comparisons with previous analysis of the Gregory-Laflamme transition we perform our analysis in the full black brane solution (obtained by replacing $d_1 \lambda' / u^6 \rightarrow (\alpha' / L^2)^2 + d_1 \lambda' / u^6$ in (5)); as the unstable mode turns out to be localized within the near-horizon region, working directly in the near-horizon geometry (5) will yield identical results.

In order to analyze the instability it is useful to lift the background (5) to M theory; this is a useful trick for simplifying the analysis, even though we are really only interested in weakly coupled IIA backgrounds in which the circle of the eleventh dimension is very small. Recall that D0-brane charge is reinterpreted as momentum around the M theory circle; consequently the M theory lift of (5) is simply obtained from (the near-horizon limit of) a toroidally compactified uncharged black 2-brane in M theory,

$$ds^2 = -f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_7^2 + dy^2 + dx^2,$$

$$f(r) = 1 - \frac{r_0^6}{r^6}, \quad (6)$$

by boosting along x , the M theory circle. This leads to

$$ds^2 = -d\tau^2 + dx^2 + \frac{1}{f(r)}dr^2 + r^2 d\Omega_7^2 + dy^2 + (1 - f(r))(\cosh \beta d\tau + \sinh \beta dx)^2. \quad (7)$$

Here y is a rescaled coordinate on our T-dual compact circle (proportional to $\tilde{\theta}$ in (5)) with period $y \equiv y + \tilde{L}$. One can determine \tilde{L} and the M theory parameters (r_0 , the Planck Mass, the radius l_{11} of the M theory circle and the boost rapidity β) such that the near-horizon limit of the background (7), reduced back to 10-d along the x direction, will be equal to (5). In particular, it is easy to verify that $\tilde{L} = (2\pi)^2 \alpha' / L$ and $u_0 = r_0 L / \alpha'$.

Fluctuations around the black brane solution (6) include a Gregory-Laflamme zero mode (carrying some momentum in the y direction) when r_0 / \tilde{L} is equal to some number $a(0)$, where one finds $a(0) \approx 0.37$. As this mode is independent of both t and x , choosing a suitable gauge, it is preserved by the boost (this argument was used in [20]). Thus, we conclude that the fluctuations around (7), and thus (5), also include a zero mode at $u_0 = (2\pi)^2 a(0)$. It is well-known that in the solution (6) this zero mode bounds a region of instability [6, 7, 8], and we will see below that this is true also for the near-horizon limit of the boosted solution (5), which becomes unstable when $u_0 < (2\pi)^2 a(0)$. Hence, in the gauge theory, the phase corresponding to the uniform black string becomes unstable below a critical temperature

$$t_{GL} = \frac{3}{4\sqrt{\pi}} \frac{(2\pi a(0))^2}{\sqrt{\lambda'}}. \quad (8)$$

In the rest of this section we explicitly demonstrate that the fluctuation spectrum of (7) includes an instability for $u_0 < (2\pi)^2 a(0)$ (and thus, in the dual theory, for temperatures lower than t_{GL}), and verify that this unstable fluctuation is supported purely within the near-horizon geometry. The perturbation of interest takes the form

$$h_{\mu\nu}(\tau, r, y) = e^{\tilde{\Omega}\tau} \Lambda_\mu^{\mu'} \Lambda_\nu^{\nu'} \left[\cos \tilde{k} y b_{\mu'\nu'} + \sin \tilde{k} y c_{\mu'\nu'} \right], \quad (9)$$

where Λ is the Lorentz transformation corresponding to the boost,

$$\begin{aligned} \tilde{\Omega} &= \frac{\Omega}{\cosh \beta}, \\ \tilde{k}^2 &= k^2 + \Omega^2 \tanh^2 \beta, \end{aligned} \quad (10)$$

and the nonzero components of b, c are

$$\begin{aligned} b_{\tau\tau} &= h_\tau(r), \quad b_{rr} = h_r(r), \quad b_{\tau r} = \Omega h_v(r), \\ b_{yy} &= \cosh^2 \alpha h_y(r), \quad b_{xx} = -\sinh^2 \alpha h_y(r), \quad b_{rx} = k \sinh \alpha h_v(r), \\ c_{ry} &= k \cosh \alpha h_v(r), \quad c_{xy} = -k \sinh \alpha \cosh \alpha h_y(r), \end{aligned} \quad (11)$$

where α is defined by $k \sinh \alpha = \Omega \tanh \beta$. Note that the perturbation has no functional dependence on the x coordinate so its reduction to IIA supergravity is straightforward, although it does involve the type IIA RR 1-form and dilaton as well as the metric.

It follows from the linearized Einstein equations that h_y obeys a simple second order eigenvalue equation in Ω^2 ,

$$\begin{aligned} h_y''(r) + p(r)h_y'(r) + q(r)h_y(r) &= \Omega^2 w(r)h_y(r), \\ p(r) &= \frac{1}{r} \left(1 + \frac{6}{f(r)} - \frac{16 k^2 r^2}{k^2 r^2 + 21 \frac{r_0^6}{r^6}} \right), \\ q(r) &= \frac{1}{r^2} \left(-\frac{k^2 r^2}{f(r)} \frac{k^2 r^2 - 27 \frac{r_0^6}{r^6}}{k^2 r^2 + 21 \frac{r_0^6}{r^6}} \right), \\ w(r) &= \frac{1}{f(r)^2}. \end{aligned} \quad (12)$$

Once h_y is determined, h_τ , h_r , and h_v may be simply derived from it. The boundary conditions are as for the original Gregory-Laflamme analysis [6].

We see that (12) is completely independent of the boost parameter β . Hence, from our knowledge of the unboosted problem, we know that for $k < k_c = 2\pi a(0)/r_0$ there exist physical modes with real positive Ω that lead to dynamical instability of (7) for all β . Of course, the precise nature of this unstable mode does depend on the boost; note that the instability exponent $\tilde{\Omega}$, the wavelength \tilde{k} , and the tensor structure – through α – are all functions of β .

As for the unboosted case, the unstable modes decay away from the horizon as $h_{\mu\nu} \sim e^{-2\pi a(\Omega)r/r_0}$, where $a(\Omega)$ is a number of order unity, and $0 < a(\Omega) < a(0)$. The translation to (5) takes r/r_0 to u/u_0 . Consequently, the unstable mode is a normalizable dynamical degree of freedom even strictly in the near-horizon limit (5) ⁴.

B. Other phases: Localized and non-uniform

Following [9, 10, 11, 12] we have argued that strongly coupled maximally supersymmetric 1 + 1 dimensional Yang-Mills theory on a circle undergoes a phase transition at a temperature of order $t \sim 1/\sqrt{\lambda'}$, since the high temperature phase becomes unstable there. We now wish to investigate how the system behaves at lower temperatures.

As we have seen, the AdS/CFT correspondence maps (the near horizon limit of) near extremal charged black solutions of IIA supergravity on $R^{8,1} \times S^1$ to phases of maximally supersymmetric Yang-Mills theory on S^1 . The thermodynamic properties of a given Yang-Mills phase are easily obtained from the Bekenstein-Hawking thermodynamics of the corresponding charged black solution. Furthermore, we have seen in the previous section that we may generate near extremal charged solutions from uncharged solutions (via an M-theory lift-boost-reduction)⁵. Hence, a complete understanding of uncharged solutions on $R^{8,1} \times S^1$ would fix the phase structure of the Yang-Mills theory under study (at strong coupling, where the supergravity approximation is valid). In this section, we review what is known about these uncharged solutions for the theory on a circle.

1. Uncharged solutions

So far three branches of uncharged solutions on spaces with an asymptotic circle have been found. These are the uniform black string (discussed in the last section), the localized black hole, and the non-uniform string, which are distinguished asymptotically by two gravitational charges [31, 32, 33, 34, 35] ⁶.

As outlined above, uniform strings on a circle are stable for masses larger than the mass of the string which possesses the Gregory-Laflamme zero mode. Below this mass they are dynamically unstable. The known non-uniform string solutions live on a branch which emerges from the uniform string branch at the marginal point. For the dimensions of interest, near the marginal point the non-uniform solutions have larger mass than the unstable uniform strings, and lower entropy than a stable uniform string with the same mass [38, 39]. In the 6-d case these solutions have been numerically constructed away from the marginal point, and these properties appear to continue to the end of the branch where the minimal sphere of the horizon pinches off [40]. However, the 10-d solutions have not been constructed so far.

Localized black holes are believed to be stable at low energies. These solutions have recently been constructed numerically in 5-d and 6-d [41, 42], and they can also be constructed in perturbation theory, expanding in the black hole mass [43, 44] ⁷.

⁴ The thermodynamics of the IIA uniform smeared near-extremal D0-brane gravity solution (5) is the same as that of the T-dual D1-brane background (1). As we have shown, the smeared D0 solution is unstable up to extremality (for large enough circle radius), whereas the T-dual D1 solution is stable away from extremality [21]. The Gubser-Mitra conjecture [22, 23, 24, 25, 26, 27] links dynamical and local thermodynamic stability. It is easy to show that if for the D0 solution one allows both the charge and the mass to vary when computing the thermodynamic stability as in [23], then this background is indeed predicted to be thermodynamically unstable up to extremality, whereas the D1 solution with fixed charge is stable. Physically, allowing the D0 charge to vary is sensible as the D0 charge can be defined as an integral over a local density on the torus, just as the mass density is local, whereas for the D1 solution a charge density cannot be constructed, and thus we think of it as a global quantity, and hence fixed. Essentially we should allow the D0 mass and the charge to be able to redistribute themselves over the torus when computing the thermodynamic stability. See also the recent [28].

⁵ Such a procedure for “charging up” solutions was previously used in [29, 30].

⁶ With the $R^{d-1} \times S^1$ asymptotics there are two gravitational charges, and interestingly the known solutions appear to be uniquely distinguished by them [36], unlike general higher dimensional black holes which violate uniqueness in terms of the asymptotic charges [37].

⁷ Analytic constructions are restricted to 4-d [45], where there are no black string solutions and hence no Gregory-Laflamme dynamics.

Kol has suggested that the black holes grow until they cannot ‘fit’ in the circle, and then they join the end of the non-uniform string branch via a cone-like topology changing solution [46] (see also [30]). Evidence supporting this elegant picture has recently emerged from the numerical solutions [47, 48, 49]. As pointed out by Kol, this picture seems rather natural when considering a Gross-Witten like eigenvalue transition.

2. Near-extremal charged solutions

As mentioned above, if we knew the 10-d uncharged solutions on a circle, we would then simply be able to translate them into near-extremal charged solutions which are relevant for the gauge theory, and to plot a phase diagram for the near extremal solutions. In appendix A we explicitly show how this would be done, by ‘charging’ an uncharged solution via the M-theory lift-boost-reduce procedure, and then examining the near extremal limit. This process would determine which solution is thermodynamically favoured at a given energy or temperature.

In Appendix B we take the first steps in implementing this programme. We use Gubser’s perturbative construction [38] to repeat Sorkin’s calculation [39] of the 10-d non-uniform strings. We then apply the M-theory charging procedure of Appendix A to determine the thermodynamic properties of near-extremal weakly non-uniform solutions. Our results demonstrate that, within the perturbative approach outlined above, near extremal non-uniform strings are thermodynamically disfavoured over the stable uniform strings, just like their uncharged counterparts. In particular, in the canonical ensemble, non-uniform solutions have a less favourable free energy than the uniform strings of the same temperature.

Unfortunately, full numerical constructions for all types of uncharged solutions in 10-d (that would extend the results of Appendix B to increasingly non-uniform strings, and would supply similar results for localized black holes) are not yet available. The equivalent constructions currently exist only in 6-d. However, we think it plausible that Kol’s picture is also true for the 10-d uncharged solutions (and for other dimensions obeying $d \leq 13$), and further that the thermodynamic ordering of the three phases is preserved by boosting. As evidence for the second part of this statement, we explicitly verify in Appendix A that the boost does preserve the order of entropy as a function of mass of the three solutions for the known 6-d solutions.

A conjectured plot of the free energy as a function of temperature for strongly coupled Yang-Mills theory, that is consistent with the above information, is shown in figure 1. As a function of the temperature, the system in the figure undergoes a single first-order phase transition, from a low temperature black hole phase to a high temperature uniform black string phase, at a temperature above the Gregory-Laflamme critical temperature (8).

We emphasize that this figure is only a guess, based largely on the results of section II A, Kol’s picture and the calculation in Appendix B. It will be possible to verify this figure (and the ensuing phase diagram) once numerical (or other) data on the properties of black holes and black strings in 10 dimensional gravity on a circle becomes available.

C. Wilson loops as order parameters

Euclidean 1 + 1-dimensional $SU(N)$ SYM on a torus has two non-contractible Wilson loops, namely the Wilson loops running around the time (τ) and space (x) circles. Consider the spatial Wilson loop,

$$P_x = P e^{i \oint dx A_x}. \quad (13)$$

According to the usual rules of T-duality, when we consider this theory as coming from D1-branes on a circle, the phases of the eigenvalues of P_x represent the positions of T-dual D0-branes on the dual circle (in a similar fashion, the eigenvalue distributions of the 8 SYM adjoint scalar zero modes represent the transverse positions of the D0-branes). Hence, the 3 potential phases discussed above – uniform, non-uniform and localized – are expected to be distinguished by the eigenvalue distribution of P_x (which is continuous in the large N limit). The uniform string corresponds to a uniform eigenvalue distribution. The non-uniform string maps to a non-uniform eigenvalue distribution that breaks translational invariance but is nowhere zero on the unit circle in the complex plane. The black hole is expected to be characterized by an eigenvalue distribution that is sharply localized on the unit circle (a distribution that is strictly zero outside an arc on the circle).

The temporal Wilson loop P_τ is expected to have a localized eigenvalue distribution for all of these phases, corresponding to the presence of a horizon [5, 15].

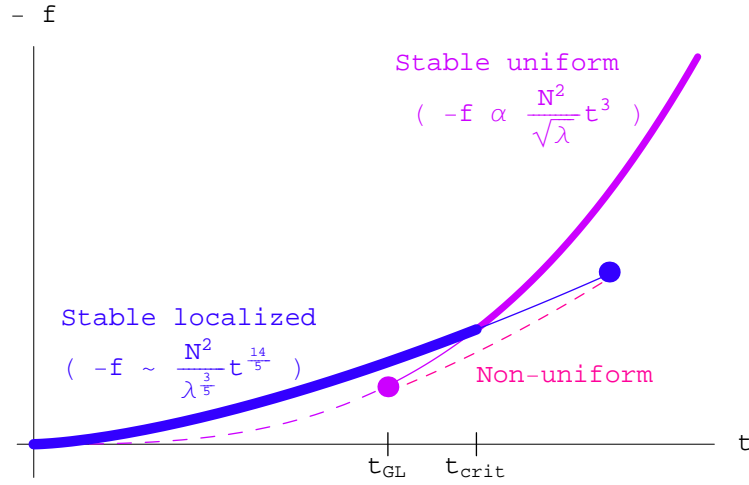


FIG. 1: An educated guess for the free energy as a function of temperature for black holes, uniform black strings, and non-uniform black strings in 10 dimensions. Appendix B shows that the non-uniform solutions have less favoured free energy than the uniform solutions near t_{GL} . Analogy with $d = 6$ indicates that the non-uniform and localized phases may meet at a topology changing solution. Dashed lines represent phases conjectured to be unstable.

III. SUPERSYMMETRIC YANG-MILLS THEORY AT WEAK COUPLING

We now turn to the study of the thermodynamics of the 1 + 1-dimensional maximally supersymmetric $SU(N)$ Yang-Mills theory at weak coupling. The Euclidean action for the SYM theory is

$$S = \frac{1}{4g_{YM}^2} \int d\tau dx \text{Tr} \left(F_{\alpha\beta}^2 + 2 \sum_I D^\alpha \phi^I D_\alpha \phi^I - \sum_{I,J} [\phi^I, \phi^J]^2 + \text{fermions} \right), \quad (14)$$

where ϕ^I are 8 adjoint scalars, x is periodic with period L , and τ , the Euclidean time, is periodic with period $\beta = L/t$. Both scalars and fermions are taken to be periodic on the spatial circle x . As usual, the fermions are anti-periodic in τ , distinguishing the temporal and spatial directions.

A. Zero mode reduction

The first approximation to this 1 + 1-dimensional theory is to reduce to zero modes on the 2-torus. Kaluza-Klein (KK) modes on both cycles of the torus are weakly coupled and can be integrated out within the window of opportunity:

temporal KK strongly coupled $t < \lambda^{1/3}$	zero mode reduction $\lambda^{1/3} < t < \frac{1}{\lambda}$	spatial KK strongly coupled $t > \frac{1}{\lambda}$
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(15)

In the intermediate regime, the theory is well described by a simple bosonic $SU(N)$ matrix integral, which has previously been studied in [50],

$$Z = \int d\psi e^{\frac{N}{4} \text{Tr}(\sum_{A,B} [\psi^A, \psi^B]^2)}, \quad (16)$$

where now the 10 traceless adjoint matrices ψ^A are composed of the zero modes of the 8 traceless scalars and the gauge potential as:

$$\{\psi^0, \psi^1, \psi^I\} \equiv \frac{1}{\beta(\lambda t)^{1/4}} \int d\tau dx \{A_\tau, A_x, \phi^I\}. \quad (17)$$

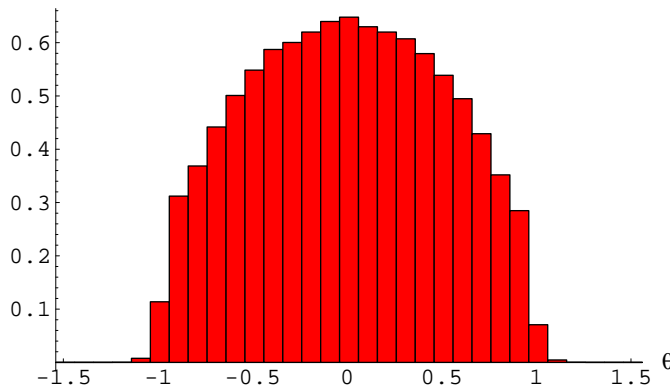


FIG. 2: Monte Carlo determination of the saddle point eigenvalue distribution of ψ^A for the matrix integral (16) with $N = 20$.

Note that no fermion zero modes survive the reduction as they are anti-periodic on the temporal circle.

This theory has a classical moduli space along which the matrices ψ^A , nine of which correspond to the spatial configuration of the D0-branes, are diagonal. While this space is noncompact, the 1-loop effective potential obtained by integrating out the off-diagonal modes in (16) is attractive; this ensures convergence of the moduli space integral. However, this perturbative calculation is only valid when the eigenvalues of ψ^I are spread out over a distance scale that is much larger than unity. At shorter separations, the off-diagonal modes become light and hence strongly coupled. It turns out that strong coupling effects stabilize the eigenvalue distribution in a saddle point that is sharply localized, with a characteristic scale of order unity. Note that this scale is independent of N , as follows from 't Hooft scaling. We have verified this claim by a Monte-Carlo integration, finding $\langle \frac{1}{N} \text{Tr}(\psi^A \psi^A) \rangle \simeq 2.5$ in the large N limit (this computation was first performed by [50]; see also [51]). We plot the eigenvalue distribution in figure 2, which shows that it is compactly supported. The spread of the eigenvalues of the ψ 's is of order one, meaning that the spread of the eigenvalues of the original matrices A_τ, A_x and ϕ^I is of order $(\lambda't)^{1/4}/L$.

B. High temperature reduction

As we have argued in the previous subsection, the $1+1$ -dimensional theory under consideration reduces to a zero mode integral in the range $\lambda'^{1/3} \ll t \ll 1/\lambda'$. It follows that the D0 branes are clumped in a spherically symmetric distribution in this range. This may be thought of as the weak coupling analog of the approximate spherical symmetry displayed by low temperature D0-brane gravity solutions on a circle.

As the temperature is increased to $\lambda't \sim 1$, the width of the zero brane distribution approaches unity and the zero branes begin to spread out over the circle. In the SYM theory, the spread of the eigenvalue distribution of A_x approaches the periodicity of this variable (arising from large gauge transformations) which is $2\pi/L$. It is natural to guess that the eigenvalue distribution undergoes a Gross-Witten like unclumping transition at a temperature of order $t \sim 1/\lambda'$. Unfortunately, the spatial KK modes become strongly coupled exactly around this temperature; consequently, this guess is difficult to verify analytically. The spatial KK modes are essential to the dynamics of the system at this temperature; they are required in order to correctly reproduce the compactness of the $SU(N)$ gauge group (the periodicity of A_x), and certainly cannot be ignored. Note, however, that temporal KK modes are utterly negligible at this temperature. Consequently, the conjectured phase transition should be well described by a $0+1$ dimensional bosonic gauge theory obtained by dimensional reduction along the temporal circle⁸. Again, the fermionic

⁸ Strictly speaking the system is well described by a $0+1$ dimensional bosonic gauge theory with an inbuilt UV cutoff at scale $\sim T$. This cutoff regulates the one-loop contribution to the cosmological constant (the only divergent graph in the theory) yielding a term in $\ln Z$ that scales like $N^2 \lambda L^3 T$. This easily calculable term is the contribution of free high energy partons to our system. It dominates the partition function in the regime $t \gg 1$ and $\lambda' \ll 1$ but makes no contribution to the eigenvalue potential, and so is of limited relevance to the analysis in this paper. See [52] for similar remarks in the context of 4 dimensional gauge theories. We thank A. Hashimoto for

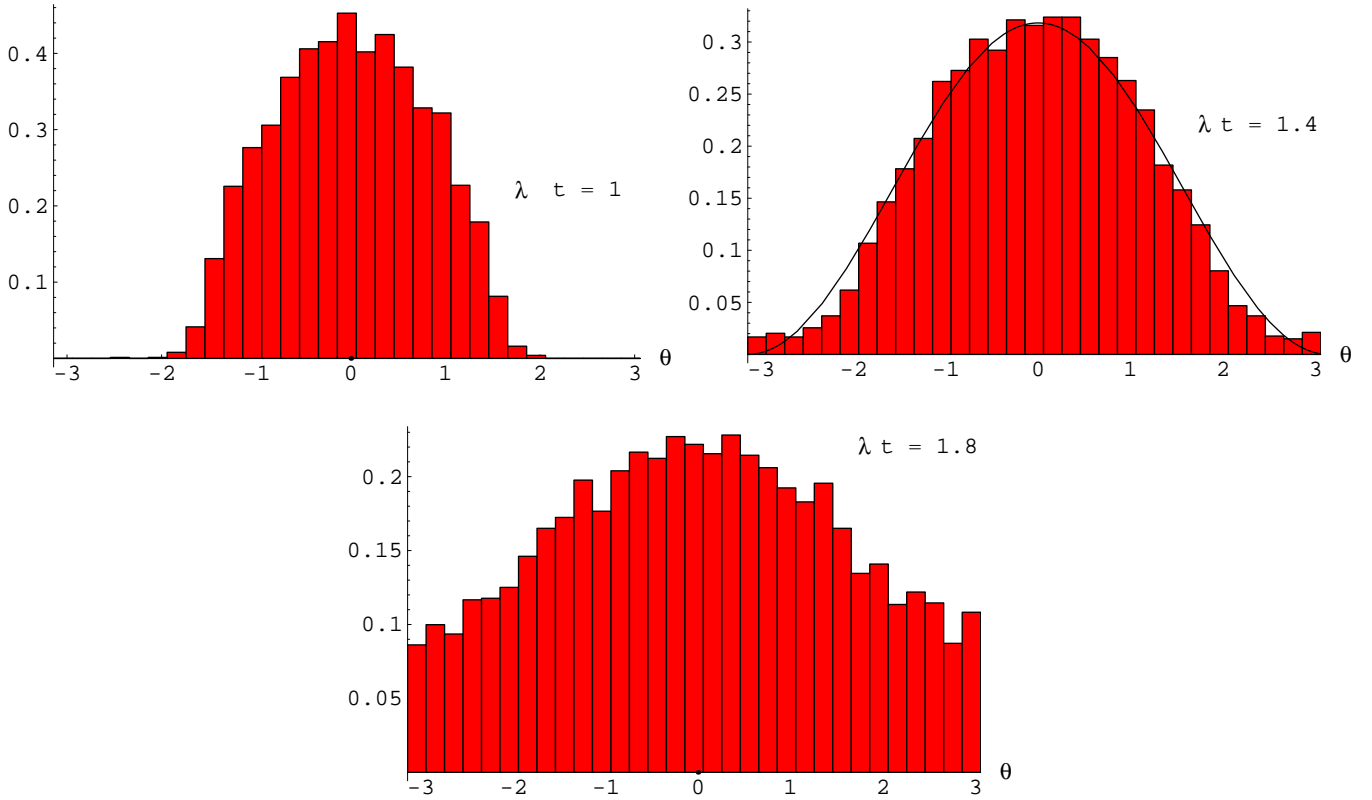


FIG. 3: Eigenvalue distribution of P_x for various values of $\lambda't$ with $N = 12$. A phase transition appears to occur for $\lambda't \simeq 1.4$, and for this value we compare with $(1 + \cos \theta)/(2\pi)$, the transition distribution for our $b = 0$ Landau-Ginzburg model discussed in section III C.

modes are anti-periodic around the temporal circle, and so may be ignored. This theory can be numerically solved using standard lattice Monte-Carlo methods. Being a one-dimensional theory, the continuum limit is rather simple to obtain, with relatively few lattice points being required (even using only 5 points gives rather accurate results, with very small corrections upon lattice refinement). The gauge dynamics in this theory are trivial and can be removed, up to the spatial Wilson loop, by choosing a gauge in which all the lattice link variables are diagonal and equal – of course, care must then be taken to implement the measure correctly.

The results of this simulation are indeed compatible with a Gross-Witten like phase transition at $\lambda't \simeq 1.4$ at large N . In figure 3 we show some typical eigenvalue distributions for the spatial Wilson loop, for couplings below, around and above the transition temperature. We clearly see that below the transition the distribution has compact support, while above it is smeared over the full angular period.

As is apparent from figure 3, for $N = 12$ the system undergoes a smooth transition between the localized phase and the non-uniform phase. However, this transition gets sharper as N is increased. This is evident by looking at the first two Fourier modes of the eigenvalue distribution, u_1 and u_2 (defined by $u_n = |\frac{1}{N} \sum_i z_{(i)}^n| = \frac{1}{N} |\text{tr}(P_x^n)|$, where $z_{(i)}$ denote the eigenvalues of P_x), shown in figures 4 and 5 as a function of $\lambda't$ for various values of N . Note that the plot of u_1 versus $\lambda't$ appears to develop a sharp jump (from $u_1 \approx 1/2$ to $u_1 = 0$ at $\lambda't \approx 1.4$) in the limit $N \rightarrow \infty$. These results strongly suggest the existence of a sharp phase transition at $\lambda't \approx 1.4$ and large N . Recall that, as discussed in section II C, the u_n are the expected order parameters for the transition, and they should vanish in a “uniform black string”-like phase. Note that the scalar zero mode also responds sharply at the transition point (figure 6).

Interestingly, at the transition point $u_1 \simeq 0.5$ and the higher Fourier modes are small. Thus it seems likely that u_1

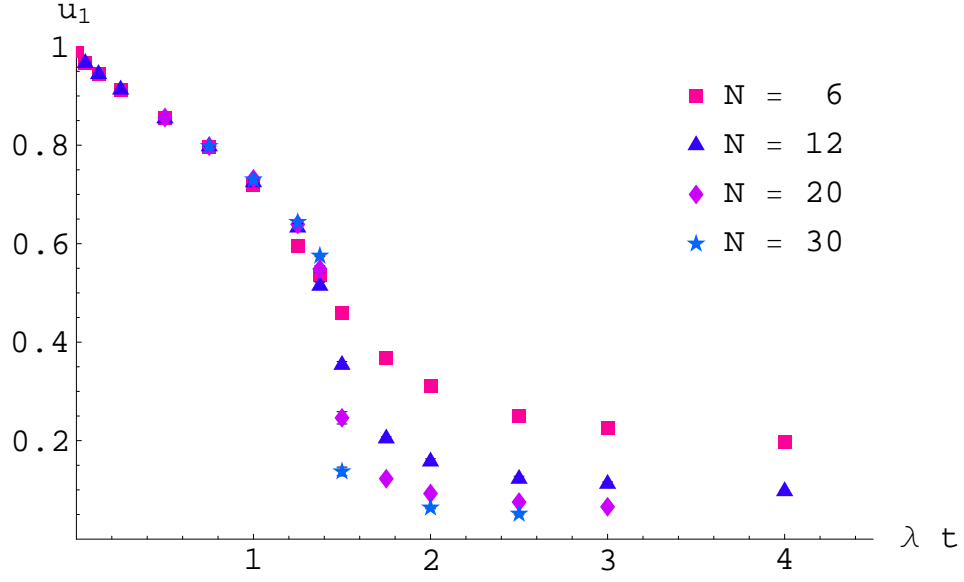


FIG. 4: The first Fourier mode u_1 of the eigenvalue distribution of P_x as a function of $\lambda't$, for various values of N . To the left of the transition at $\lambda't \simeq 1.4$, the values remain approximately invariant as N increases. To the right, u_1 decreases consistent with going to zero in the large N limit as $1/N$. Statistical error bars are smaller than the plot symbols.

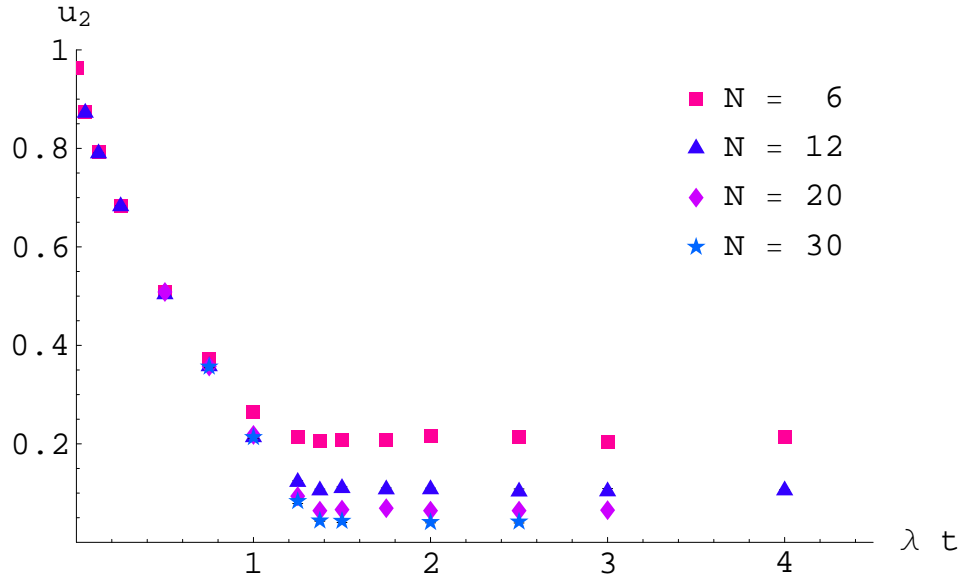


FIG. 5: The second Fourier mode u_2 of the eigenvalue distribution of P_x as a function of $\lambda't$ for various values of N . For large N the gradient appears to become discontinuous at the transition.

is the dominant order parameter for the transition, so that the first Fourier component of the eigenvalue distribution gives the most relevant low mass mode near the transition.

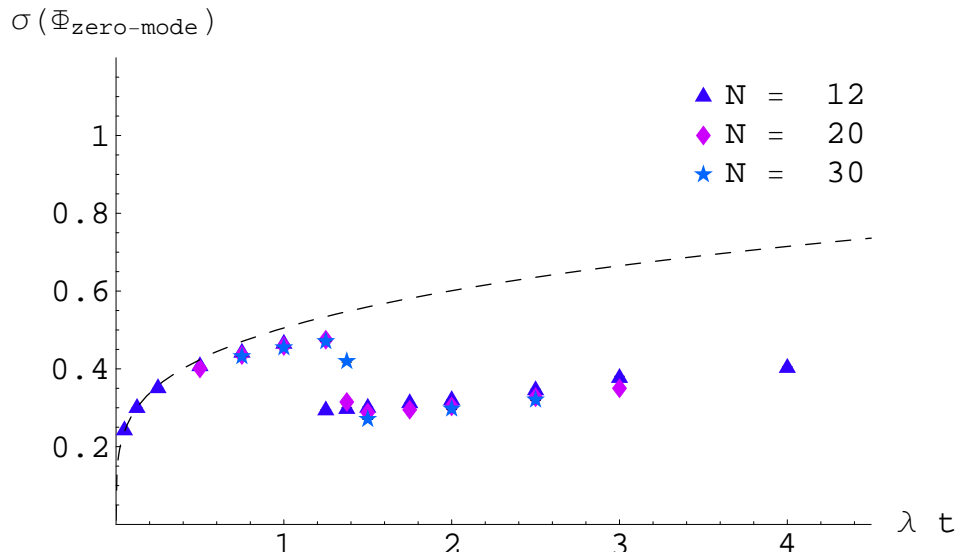


FIG. 6: The scalar zero mode eigenvalue distribution width as a function of $\lambda't$, for various values of N . The dotted line gives the theoretical result from the $0+0$ dimensional matrix integral, which (as expected) reproduces the behaviour at small $\lambda't$.

C. Landau-Ginzburg Analysis

As we have noted above, the order parameter of our system appears to jump discontinuously at $\lambda't \simeq 1.4$ ⁹. It is tempting to interpret this observation as evidence for a first order phase transition. However, it is also possible that, instead, the system undergoes two continuous phase transitions, the first at $u_1 \sim 0.5$ and at $\lambda't \approx 1.4$, and the second at $u_1 = 0$ and $\lambda't$ a little larger than 1.4. This possibility is not as outlandish as it might first seem, as we now explain.

We have attempted to fit the data from our Monte Carlo simulation to the predictions from the following Landau-Ginzburg model, with a and b smooth functions of $\lambda't$,

$$Z_{MM} = \int dU e^{a|\text{Tr}(U)|^2 + b|\text{Tr}(U)|^4/N^2}. \quad (18)$$

Identifying U with P_x (so that $u_n = \frac{1}{N}|\text{Tr}(U^n)|$) the surprise is firstly that this works remarkably well, and secondly that b is small. Let us set b to zero. In this case the model (18) exhibits a weakly first order phase transition at $a = 1$ [14, 15]. Figure 7 shows the value of a as a function of $\lambda't$ found by fitting the values of u_1 measured from the $N = 6$ lattice simulations to the values coming from the matrix model for $N = 6$ (with $b = 0$). Note that this curve continues smoothly through the transition. For this parameterization of a , in figure 8 we then plot the predictions for u_1 and u_2 from the model (18), now with $N = 12$, compared to the actual data (also with $N = 12$). We find excellent agreement within the error bars of the lattice data, and in fact find a similar agreement also for $N = 20, 30$.

It is striking that our Landau-Ginzburg model fits the Monte Carlo data so well for $b \approx 0$. In order to explain the significance of this observation, we will now digress to review the solution of the model (18) at large N . As explained in section 6 of [15], this model can be exactly solved in terms of the Gross-Witten matrix integral. When b is small and negative, this model undergoes a single first order phase transition from a black hole like clumped eigenvalue phase for $a > 1 - |b|/4$ to a uniform eigenvalue distribution for $a < 1 - |b|/4$. The order parameter u_1 jumps discontinuously from $u_1 \sim \frac{1}{2}$ to $u_1 = 0$ on going through this transition. When b is small and positive, the system undergoes two continuous phase transitions : from a clumped eigenvalue phase (for $a > 1 + b/4$) to a non-uniform string phase (for

⁹ Thus our results disagree with the ‘mean field’ prediction of a single continuous phase transition in the same theory, reported in [53].

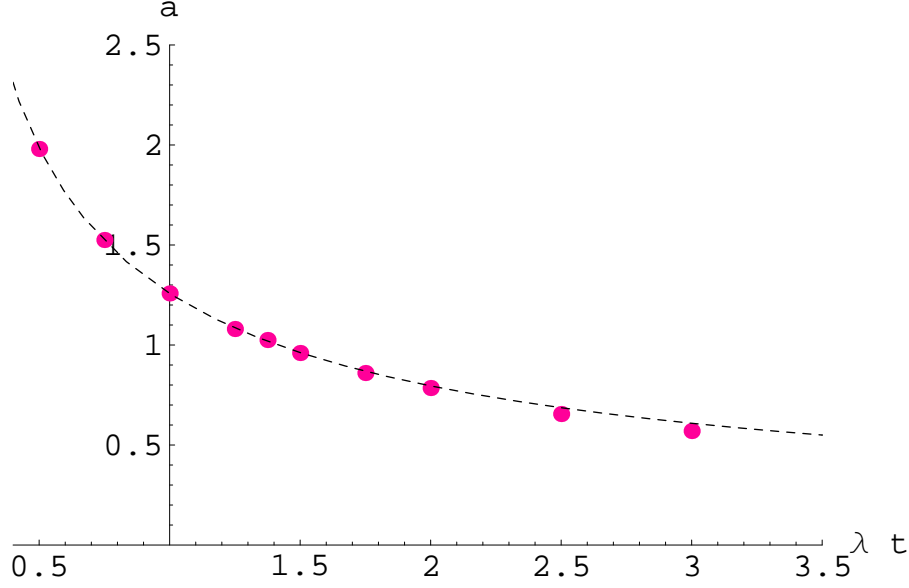


FIG. 7: The data points show the values of $a(\lambda't)$ obtained by fitting the matrix model with $b = 0$ to the $0 + 1$ -dimensional lattice data. The value of a is determined by fitting for u_1 with $N = 6$. Note that the behaviour of a appears smooth across the transition point, indicating that the matrix model correctly reproduces the transition behaviour of P_x . The dashed line is the curve $1.3/(\lambda't)^{2/3}$, which appears to give a reasonable fit to the data points over this range.

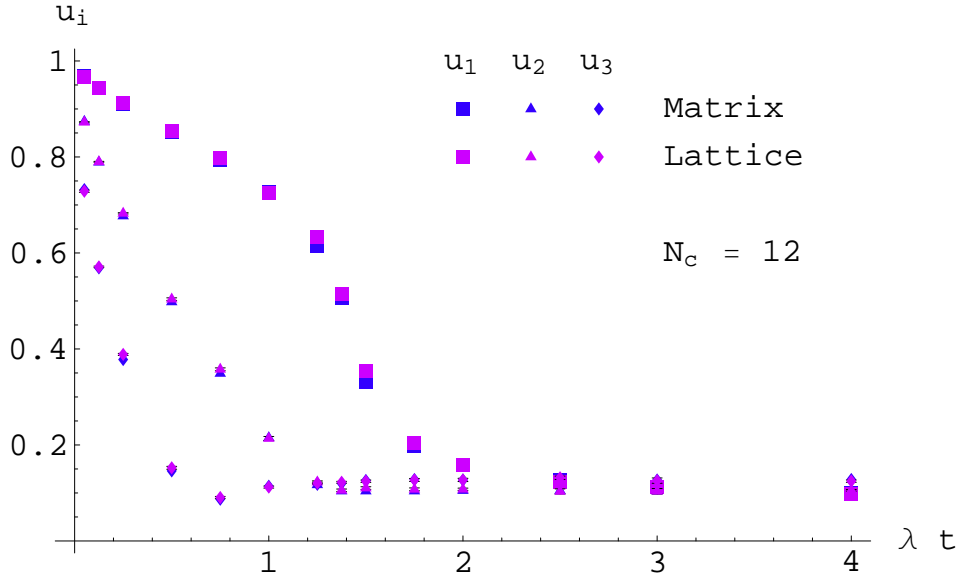


FIG. 8: Comparison of $0 + 1$ -dimensional Monte Carlo data and Landau-Ginzburg results for the matrix model with $b = 0$, with a determined as in the previous figure. We clearly see excellent agreement for the various moments $u_{1,2,3}$ for $N = 12$ as plotted here. A similar good agreement is observed for $N = 20, 30$.

$1 < a < 1 + b/4$) to a uniform string phase (for $a < 1$). As a varies between 1 and $1 + b/4$, the order parameter u_1 evolves continuously from $\frac{1}{2}$ to zero. This continuous variation approaches a discontinuous jump as $b \rightarrow 0$.

Our numerics are unable to distinguish very small values of b , either positive or negative, from $b = 0$ (hence a single first order phase transition from two closely separated continuous transitions). Our system clearly lies very near the cusp that separates these two possibilities. It is even possible that b is exactly zero in the weak coupling limit for some good reason that we have not yet understood (a similar situation is known to arise in a related context [15], and

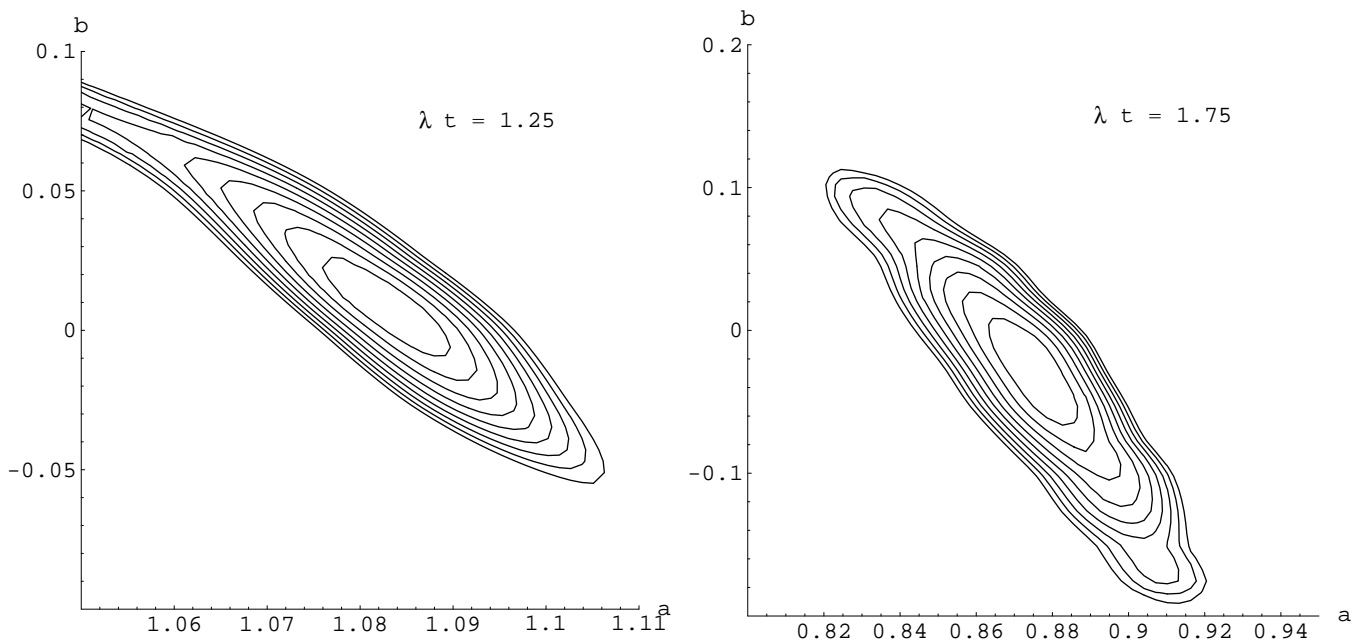


FIG. 9: Plots of least squares fit function L in the (a, b) plane, for values of $\lambda't$ just below and just above the transition point. For each plot the contour interval is chosen to equal the minimum value of L , and hence the contours give an indication of decline of goodness of fit away from the minimum. For both plots we see the most likely value of b is numerically small.

a similar surprising agreement to the simple $b = 0$ matrix model was recently observed in [54]).

We now attempt to quantify the smallness of b . Using the lattice Monte-Carlo data for various $N = 6, 12, 20$, we compare u_i from the matrix model with the lattice calculation. Note that even for the best values we do not expect a perfect fit as our Landau-Ginzburg model can be further refined by adding in higher order terms in $|\text{Tr}(U)|$, as well as terms depending on traces of higher powers of U . We use a least squares fit to test the lattice data u_i and the Landau-Ginzburg matrix model predictions $u_i^{\text{MM}(a,b)}$ for particular (a, b) , defining the goodness of fit function L to be minimized as,

$$L(a, b) = \sum_{i=1,2} \sum_{N=6,12,20} \left(u_i^{\text{MM}(a,b)}(N) - u_i(N) \right)^2. \quad (19)$$

In figure 9 we plot contours of L in the a, b plane, calculated for two values of $\lambda't$ on either side of the transition point. In each case contour intervals are chosen to equal the minimum value of L , and thus indicate goodness of fit. We see that both plots give a most likely b close to zero.

It is noteworthy that, for any nonzero b , the Landau-Ginzburg model (18) possesses exactly 3 saddle points [15] that match perfectly with the 3 gravitational phases – the uniform string, the non-uniform string and the black hole – that we have described in Section II. When $b < 0$, the non-uniform string is never thermodynamically favoured, and figure 1 qualitatively captures the thermodynamics of our system. On the other hand, when $b > 0$, the non-uniform string phase is the thermodynamically favoured solution throughout its existence; in fact in such a case it is the (thermodynamic) end point of the Gregory-Laflamme transition. This is similar to the expected behaviour of the Gregory-Laflamme transition when $d \geq 14$ [39].

IV. SUMMARY

Holography predicts that 1+1-dimensional maximally supersymmetric Yang-Mills theory on a circle of circumference L undergoes a first order phase transition at strong 't Hooft coupling at a critical temperature equal to or larger than $T_{GL} = 3(2\pi a(0))^2/4\pi^{1/2}L\sqrt{\lambda}$, where $a(0) \approx 0.37$. The order parameter for this phase transition is the eigenvalue

distribution (uniform at high temperature, clumped at low temperature) of the Wilson loop along the spatial circle. For the dual supergravity theory, the relevant dynamics is the Gregory-Laflamme instability.

At weak coupling, we have found good evidence for a sharp phase transition between identical phases at a temperature $T_c \simeq 1.4/\lambda L^3$. We conjecture that a similar transition occurs at all values of the 't Hooft coupling. This picture suggests that the dynamics of compactified horizons probed here, essentially governed by the Gregory-Laflamme instability, is stable to α' corrections. Note that a weak coupling behaviour $t_c \sim 1/\lambda'$ is replaced at strong coupling by $t_c \sim 1/\sqrt{\lambda'}$; similar changes in the dependence on the coupling constant have been observed in many other cases of theories with holographic duals, for example in the quark-anti-quark potential in the $\mathcal{N} = 4$ SYM theory [55, 56].

We discovered that at weak coupling the behaviour of the theory near the transition is very well approximated by a simple Landau-Ginzburg model (18). It is intriguing that (18) is also the Landau-Ginzburg model that governs the dynamics of deconfinement transitions in weakly coupled Yang-Mills theories [15]. In fact, viewed from the point of view of the $0 + 1$ dimensional Euclidean bosonic gauge theory we analyzed above, the transition we have numerically studied above is precisely the deconfinement transition. Thus, the Gregory-Laflamme and deconfinement phase transitions appear to lie in the same “universality class”.

Our Landau-Ginzburg model has precisely 3 distinct phases – the uniform, non-uniform and localized phases – and these agree with conjectured and known results in the gravity. Thus, it would be tempting to extrapolate to strong coupling and conclude that these 3 solution branches (including also their reidentifications with larger periods, for instance a single black hole solution suitably scaled and reidentified to give a solution for n black holes with the same compactification radius) exhaust the static gravity solutions, and that there are no more solution branches to be found. To investigate this from the gravity side appears to be a very difficult problem.

In order to gain more insight into the SYM phase diagram, it is obviously important to determine the order of the transition at weak coupling and to understand why it lies on, or so near to, the cusp between first and second order.

While we have focussed on $1 + 1$ dimensional Yang-Mills in this letter, the analysis of section 2 may trivially be extended to relate the behaviour of D0-branes on T^p to the thermodynamics of maximally supersymmetric $p + 1$ dimensional Yang Mills on a p -torus, at least for $p \leq 3$. It would be interesting to verify that these gauge theories also undergo Gregory-Laflamme like phase transitions¹⁰.

In this note we have studied the theory at finite temperature, and we have not attempted to address issues concerning the dynamical end-point of the Gregory-Laflamme type transition that we have studied [59]. Horowitz and Maeda [60, 61] have argued that, within general relativity, a black string like phase is unable to decay to a black hole phase in finite time. In this paper, we have identified the uniform string with a uniform eigenvalue distribution of a Wilson line operator; the black hole is a clumped eigenvalue distribution of the same operator. A cursory analysis reveals no barrier for a dynamical transition between these two phases. It would certainly be interesting to understand this better.

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¹⁰ See [57, 58] for related work in pure Yang-Mills theory.

work of TW was supported by the David and Lucille Packard Foundation, Grant Number 2000-13869A.

Appendix A

In this appendix we demonstrate how to map results about general 10-d static pure gravity solutions into results about SYM energy and entropy, by adding charge to the solutions, taking the decoupling limit and translating to SYM variables, as discussed in section II. Once the relevant solutions are computed this shows explicitly how to map them into predictions for phases of the SYM theory. Our discussion is phrased in arbitrary dimension, even though the correspondence to SYM is only valid in 10-d. This allows us to examine also the 6-d gravity example where detailed numerical solutions have been computed, and to make analogies with our 10-d case of interest.

Let us take a branch of static uncharged d -dimensional solutions which asymptote to $R^{d-2,1} \times S^1$. The metrics can be written in the general form

$$d\tilde{s}_{(d)}^2 = -A^2 d\tau^2 + V^2 (dr^2 + r^2 d\Omega_{d-3}^2) + B^2 dy^2, \quad (20)$$

where y is the compact circle coordinate ($y \equiv y + L$), and A, V, B are functions of r, y which go to unity at large r . We are interested in black solutions for which A vanishes at the horizon. We follow the 10-d prescription of section II for generating charged solutions from (20) : we add an extra periodic coordinate x , boost in this direction with parameter β and then dimensionally reduce on x . This yields a new Einstein frame metric,

$$ds_{(d)}^2 = \phi^{+\frac{2}{d-2}} \left[-\frac{A^2}{(\cosh^2 \beta - A^2 \sinh^2 \beta)} d\tau^2 + V^2 (dr^2 + r^2 d\Omega_{d-3}^2) + B^2 dy^2 \right], \quad (21)$$

where ϕ (the size of the x -direction) becomes a non-trivial dilaton in the supergravity,

$$\phi^2 = \cosh^2 \beta - A^2 \sinh^2 \beta, \quad (22)$$

and now the solution is also charged under a 1-form gauge potential coming from the $g_{\tau x}$ component of the metric (analogous to the RR 1-form potential in the 10-d case of section II)¹¹. It is easy to see that the surface gravity at the horizon, and hence the temperature, is simply redshifted by a factor of $\cosh \beta$. In addition, since A vanishes at the horizon, the entropy is simply scaled by this same factor.

Asymptotically, we can always expand the metric as

$$\begin{aligned} A &= 1 - a(\xi) \left(\frac{L}{r} \right)^{d-4} + \dots \\ B &= 1 + b(\xi) \left(\frac{L}{r} \right)^{d-4} + \dots \end{aligned} \quad (23)$$

with a, b essentially giving the two asymptotic charges of the gravity solution. The dimensionless variable ξ parameterizes a branch of solutions of the form (20) at fixed L . Using a and b we can calculate the ADM mass of the solutions (21),

$$M = \frac{L^{d-3}}{16\pi G} \Omega_{d-3} (2(d-3)a - 2b + 2(d-4)a \sinh^2 \beta), \quad (24)$$

and their charge

$$Q = \frac{L^{d-3}}{16\pi G} \Omega_{d-3} ((d-4)a \sinh 2\beta). \quad (25)$$

The entropy and temperature of (21) are given by

$$S = \frac{L^{d-2}}{4G} \cosh \beta \Omega_{d-3} s(\xi), \quad T = \frac{1}{L \cosh \beta} t(\xi), \quad (26)$$

¹¹ See also [28] for an analogous transform.

for some functions s, t which again depend only on ξ (and not on β).

Note that for fixed asymptotic circle length L , the new solutions obey the first law of black hole thermodynamics,

$$dM(\xi, \beta) = T(\xi, \beta)dS(\xi, \beta) + \mu(\xi, \beta)dQ(\xi, \beta) \quad (27)$$

with $\mu = \tanh \beta$, provided that the uncharged solution branch (20) obeys

$$dM_0(\xi) = T_0(\xi)dS_0(\xi), \quad (28)$$

where M_0, T_0, S_0 are the thermodynamic quantities for the uncharged solution (with $\beta = 0$). Note that in order to show this, one needs to use the relation

$$\frac{L^{d-3}}{16\pi G}\Omega_{d-3}(2(d-4)a) = T_0S_0, \quad (29)$$

which can be derived from [34, 35].

Next, we wish to take the near-extremal limit by taking β to infinity. We define the energy above extremality $\mathcal{E} = M - Q$, and the ‘reduced entropy’ $\mathcal{S} = S/\sqrt{Q}$. These quantities remain finite in the infinite β limit, even though the quantities M, Q , and S diverge (for fixed L, G). We may write them as

$$\mathcal{E}(\beta \rightarrow \infty) = p(\xi)\frac{L^{d-3}}{G}, \quad \mathcal{S}(\beta \rightarrow \infty) = q(\xi)\sqrt{\frac{L^{d-1}}{G}}, \quad (30)$$

where the infinite β limit defines the quantities $p(\xi), q(\xi)$, giving

$$p(\xi) = \frac{\Omega_{d-3}}{16\pi}((d-2)a(\xi) - 2b(\xi)), \quad q(\xi) = \sqrt{\pi\Omega_{d-3}}\frac{s(\xi)}{\sqrt{2(d-4)a(\xi)}}. \quad (31)$$

As in M(atrix) theory [62], in the 10-d case the infinite boost limit is equivalent to a decoupled SYM theory, and in fact the infinite boost limit is closely related to the near-horizon decoupling limit discussed in section II (as explained most clearly in [63]). After translating the quantities above to the SYM theory one finds that the SYM (dimensionless) energy ϵ and entropy σ are simply functionally related to p and q by

$$\epsilon \propto \frac{N^2}{\lambda'^2}p(\xi), \quad \sigma \propto \frac{N^2}{\lambda'^{3/2}}q(\xi). \quad (32)$$

Since the factors of N and λ' are the same for all solutions of the form (20), this shows that in order to analyze which phase is preferred in the SYM theory, we do not have to discuss the full decoupling limit of the near extremal solutions, but instead we can just consider the quantities \mathcal{E}, \mathcal{S} . Their construction from M, S and Q gives the functions $p(\xi), q(\xi)$ in the simpler infinite β limit, and this is related by (32) to the behaviour of the SYM energy and entropy (as functions of ξ) in the actual decoupling limit.

As an example of the formalism of this section, the d -dimensional uncharged uniform black string metric is

$$ds^2 = -(1 - \left(\frac{r_0}{r}\right)^{d-4})d\tau^2 + \frac{dr^2}{1 - \left(\frac{r_0}{r}\right)^{d-4}} + r^2 d\Omega_{d-3}^2 + dy^2. \quad (33)$$

Putting this in the form of (20) and applying the transformation to the near extremal charged string gives

$$p(\xi) = \frac{\Omega_{d-3}}{16\pi}\frac{d-2}{2}\xi^{d-4}, \quad q(\xi) = \sqrt{\frac{\pi\Omega_{d-3}}{d-4}}\xi^{d-2}, \quad (34)$$

where for this branch we have defined $\xi = r_0/L$. For the localized solution with $r_0 \ll L$ we may simply approximate the unboosted starting metric by the d -dimensional Schwarzschild solution, and in this limit we find that p and q are given by similar expressions to (34) but with d replaced by $d+1$.

In 6-d numerical solutions for the uncharged non-uniform string and black hole branch on a compact circle have been found. In figure 10 we plot the entropy as a function of mass for these solutions, with the y circle having unit length. We may then ask whether this diagram remains qualitatively similar when charge is added as discussed above, and the near-extremal limit is taken. Of course, since these solutions are not in 10-d we cannot use these transformed near extremal solutions to directly make statements about strongly coupled YM theory.

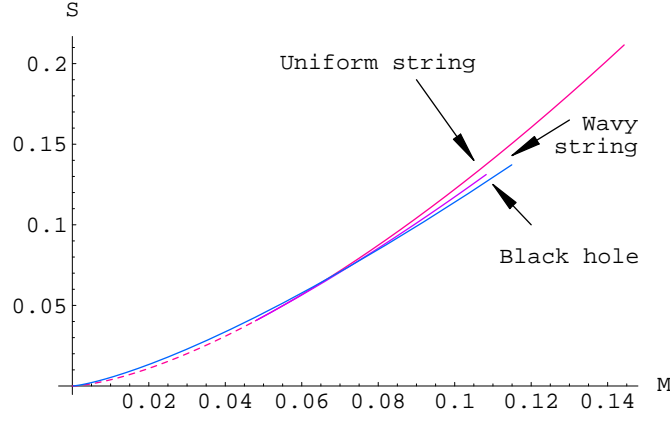


FIG. 10: Plot of entropy as a function of mass for the static uncharged black hole, non-uniform and uniform strings in $d = 6$. The results are taken from [41], where the black hole branch is only partially constructed, and over this range behaves very similarly to a 6-d Schwarzschild solution. Recent calculations indicate that the black hole branch turns around and connects to the non-uniform string branch [49].

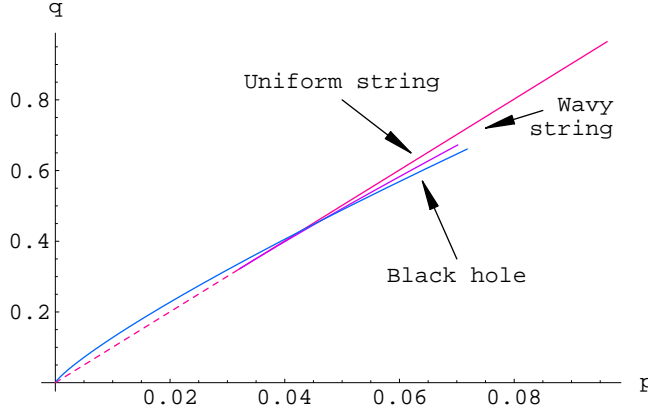


FIG. 11: Plot of p, q for the near-extremal solutions obtained by transforming the 6-d uncharged static solutions. In $d = 10$ this would give the relevant backgrounds dual to SYM. Here in $d = 6$ there is no dual, but we think of the functions p, q as analogous to energy and entropy, as in the $d = 10$ case. Note then that the entropy-mass ordering of the solutions remains the same as for the uncharged solutions of figure 10, even though a priori this is not guaranteed.

In figure 11 we plot q as a function of p for the same solutions (in the 10-d case this would give σ and ϵ in the SYM). We see that the ordering of the solutions appears to be preserved when taking this near extremal limit transform of the original uncharged solutions. We expect a similar result to hold also for 10-d, where the solutions can be related to YM phases. In the following appendix we show that in 10-d near the point where the non-uniform solutions emerge from the uniform branch, the same qualitative behaviour is indeed found in the near-extremal charged limit as for these 6-d solutions.

Appendix B

The behaviour of static uncharged non-uniform strings near the critical uniform string solution can be computed using Gubser's method [38]. Results in 10-d were recently obtained by Sorkin [39], who showed that for $d \leq 13$ the non-uniform strings have lower entropy than the uniform ones of the same mass. We now repeat these calculations, and then use the results from the previous appendix in order to obtain the behaviour of the 10-d near extremal non-uniform charged solutions relevant for this paper.

Firstly, repeating Sorkin's analysis and using his notation for brevity, perturbing to third order about a unit horizon

radius ($r_0 = 1$) uncharged uniform string we find

$$\begin{aligned} k_0 &= 2.30, & k_1 &= 0. \\ dT/T &= -0.85\lambda^2 + O(\lambda^4), & A_{inf} &= -4.19, \\ dS/S &= 9.28\lambda^2 + O(\lambda^4), & B_{inf} &= 1.66, \end{aligned} \quad (35)$$

where $dT/T, dS/S$ give the fractional difference of T, S for a non-uniform string with non-uniformness λ (which we can take to define ξ for this branch) compared to the critical uniform string $\lambda = 0$. In Sorkin's notation A_{inf}, B_{inf} give the asymptotics and can be easily related to our a, b . As stated above, for these values we have used the gauge condition $k_1 = 0$. This yields

$$\frac{S_{\text{non-uniform}}}{S_{\text{uniform}}} = 1 - 2.2\lambda^4 + O(\lambda^6), \quad (36)$$

where both solutions have the same mass, and hence at least near the critical uniform solution, as in 6-d, the uncharged uniform strings have higher entropy than the non-uniform strings.

Next, we wish to understand this behaviour for the charged limit outlined in the previous appendix. Again we consider the gravitational quantities \mathcal{E}, \mathcal{S} which, as in the previous appendix, are simply proportional to the YM ϵ, σ in the infinite β limit,

$$\mathcal{E} = M - Q, \quad \mathcal{S} = \frac{1}{\sqrt{Q}}S, \quad \mathcal{T} = \sqrt{Q}T, \quad (37)$$

and we also define \mathcal{T} , the ‘reduced’ temperature, which again is finite in the infinite β limit, where it is proportional to the YM temperature.

From now on all quantities are given for $\beta \rightarrow \infty$. We may then use our previous appendix results to find

$$\begin{aligned} d\mathcal{E}/\mathcal{E} &= 7.6\lambda^2 + O(\lambda^4), \\ d\mathcal{S}/\mathcal{S} &= 5.1\lambda^2 + O(\lambda^4), \\ d\mathcal{T}/\mathcal{T} &= 3.3\lambda^2 + O(\lambda^4), \end{aligned} \quad (38)$$

where again these denote the fractional difference from the critical uniform string. For infinite β , the first law (27) implies that

$$d\mathcal{E} = \mathcal{T}d\mathcal{S} \quad (39)$$

(using equation (29)). Since the uniform and non-uniform branches merge at the marginal solution, and hence have the same \mathcal{T} there, the ratio $\mathcal{S}_{\text{non-uniform}}/\mathcal{S}_{\text{uniform}}$, for the same energy \mathcal{E} , must also go as λ^4 , with the first law implying that the potential leading λ^2 term vanishes. However, as Gubser showed, fortunately we may use the first law to compute this difference without resorting to higher order perturbation theory. In fact, we find

$$\frac{\mathcal{S}_{\text{non-uniform}}}{\mathcal{S}_{\text{uniform}}} = 1 - 2.0\lambda^4 + O(\lambda^6), \quad (40)$$

where the solutions have the same ‘reduced’ energy \mathcal{E} .

Thus, finally, we confirm that as for the static uncharged case (36), also in the $\beta \rightarrow \infty$ limit the 10-d non-uniform solutions have lower entropy \mathcal{S} for the same energy \mathcal{E} . Since (by design) in this limit these quantities are proportional to the dual YM entropy and energy, this predicts that the same is true in the YM theory at strong coupling.

As we are interested in YM thermodynamics in which we fix the temperature rather than the energy, we may simply extend this calculation to compute

$$\frac{\mathcal{F}_{\text{non-uniform}}}{\mathcal{F}_{\text{uniform}}} = 1 - 8.0\lambda^4 + O(\lambda^6), \quad (41)$$

where $\mathcal{F} = \mathcal{E} - \mathcal{T}\mathcal{S}$ is the ‘free energy’ of the solution (which is again proportional to the YM free energy), and the two solutions which are compared have the same \mathcal{T} . This shows that the non-uniform phase has a higher free energy than the uniform one (since $\mathcal{F}_{\text{uniform}}$ is negative) at fixed temperature near the marginal point, and is thus thermodynamically disfavoured. We use this information to aid in our construction of a likely phase diagram for the YM theory in figure 1, and to argue that the strongly coupled YM theory goes through a first order phase transition.

Appendix C

The system we have investigated in this paper – $1 + 1$ dimensional maximally supersymmetric Yang-Mills theory on a circle – has been previously studied in different contexts. In this appendix we describe how our results fit in with those of previous investigations.

A. Reduction to the symmetric product CFT at very strong coupling

In the limit of very strong coupling, $\lambda' \gg N^2$, the SYM theory (14) flows, at low enough energies, to a $1 + 1$ dimensional superconformal field theory which is a sigma model on the target space $(R^8)^N/S_N$ where S_N is the permutation group on N objects [20, 64, 65, 66, 67]¹². The fermions in (14) are periodic around the spatial circle; as a consequence, the CFT is in the Ramond sector, and hence is in the ‘long string phase’ [68] at any nonzero temperature. This implies that, in the parent gauge theory, the spatial holonomy is the shift matrix, whose eigenvalues are uniformly distributed over the circle. This is consistent with our results; as we have described (see (8)), at strong coupling in the ’t Hooft limit (14) is in the uniform string phase for $t \gg \frac{1}{\sqrt{\lambda'}}$, i.e. at every temperature in the limit $\lambda' \rightarrow \infty$ considered here.

If we instead study (14) with fermions antiperiodic around the spatial circle, then it would flow, at very strong coupling, to the symmetric product CFT in the Neveu-Schwarz sector. This CFT, in the large N limit, undergoes a phase transition between a low temperature short string phase and a high temperature long string phase at exactly $t = 1$ [68, 69, 70]. In the short string phase, the spatial holonomy of the parent gauge theory is peaked around the identity matrix so this corresponds to our black hole phase. We thus conclude that (14) with antiperiodic fermions and at very strong coupling undergoes a first order black hole-black string transition at $t = 1$. This is consistent with results to appear [19].

B. Relation to Matrix String Theory

Another interesting limit of the SYM theory (14) is the limit of large N with constant g_{YM} , which appears in Matrix string theory [62, 66, 67]. When we lift the background (5) to M theory as described in section II A, it is interpreted as the near-horizon limit of an M theory configuration (with the eleven dimensional Planck scale $M_{11} \propto (L/g_{YM}^2)^{1/3}/\alpha'$) where we have N units of momentum on the circle of the eleventh dimension, smeared along a transverse circle (say, the ninth dimension). If we now reduce this background back to type IIA string theory along the ninth dimension rather than along the eleventh dimension, we find a type IIA configuration with a constant dilaton (with the string coupling proportional to $1/g_{YM}L$, and the string tension proportional to $1/(g_{YM}\alpha')^2$), and still involving N units of momentum around a compact circle, but now with no additional circles. As described, for instance, in [63], the near-horizon limit of this configuration can be identified with the DLCQ limit (or the infinite momentum frame) which is relevant to Matrix string theory.

Note that, as in the previous subsection, in the matrix string limit we have $\lambda' \rightarrow \infty$, and then (for any finite temperature in the large N limit) (14) is always in the uniform string phase and undergoes no transitions¹³.

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- [1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.
 - [2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. **B428**, 105 (1998), hep-th/9802109.
 - [3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.
 - [4] S. W. Hawking and D. N. Page, Commun. Math. Phys. **87**, 577 (1983).
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¹² The condition $\lambda' \gg N^2$ is required to flow to the orbifold CFT at energies or temperatures of order $1/L$. Outside the ’t Hooft limit one is sometimes interested in energies or temperatures scaling as $1/NL$, and for these to be well-described by the orbifold CFT it is enough to require $\lambda' \gg 1$.

¹³ For a discussion of matrix string thermodynamics, see for example [71].

- [5] E. Witten, *Adv. Theor. Math. Phys.* **2**, 505 (1998), hep-th/9803131.
- [6] R. Gregory and R. Laflamme, *Phys. Rev. Lett.* **70**, 2837 (1993), hep-th/9301052.
- [7] R. Gregory and R. Laflamme, *Nucl. Phys.* **B428**, 399 (1994), hep-th/9404071.
- [8] R. Gregory and R. Laflamme, *Phys. Rev.* **D51**, 305 (1995), hep-th/9410050.
- [9] L. Susskind (1998), prepared for ICTP Conference on Super Five Brane Physics in 5+1 Dimensions, Trieste, Italy, 1-3 Apr 1998, hep-th/9805115.
- [10] J. L. F. Barbon, I. I. Kogan, and E. Rabinovici, *Nucl. Phys.* **B544**, 104 (1999), hep-th/9809033.
- [11] M. Li, E. J. Martinec, and V. Sahakian, *Phys. Rev.* **D59**, 044035 (1999), hep-th/9809061.
- [12] E. J. Martinec and V. Sahakian, *Phys. Rev.* **D59**, 124005 (1999), hep-th/9810224.
- [13] L. Fidkowski and S. Shenker (2004), hep-th/0406086.
- [14] B. Sundborg, *Nucl. Phys.* **B573**, 349 (2000), hep-th/9908001.
- [15] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk (2003), hep-th/0310285.
- [16] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, and M. Van Raamsdonk (To appear).
- [17] D. J. Gross and E. Witten, *Phys. Rev.* **D21**, 446 (1980).
- [18] S. Wadia (1979), preprint EFI-79/44-CHICAGO.
- [19] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, M. Van Raamsdonk, and T. Wiseman (To appear).
- [20] N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, *Phys. Rev.* **D58**, 046004 (1998), hep-th/9802042.
- [21] S. Gubser and A. Ozakin, *JHEP* **05**, 010 (2003), hep-th/0301002.
- [22] S. S. Gubser and I. Mitra (2000), hep-th/0009126.
- [23] S. S. Gubser and I. Mitra, *JHEP* **08**, 018 (2001), hep-th/0011127.
- [24] H. Reall, *Phys. Rev.* **D64**, 044005 (2001), hep-th/0104071.
- [25] J. P. Gregory and S. F. Ross, *Phys. Rev.* **D64**, 124006 (2001), hep-th/0106220.
- [26] V. Hubeny and M. Rangamani, *JHEP* **05**, 027 (2002), hep-th/0202189.
- [27] S. A. Hartnoll (2004), hep-th/0403281.
- [28] P. Bostock and S. F. Ross (2004), hep-th/0405026.
- [29] S. F. Hassan and A. Sen, *Nucl. Phys.* **B375**, 103 (1992), hep-th/9109038.
- [30] T. Harmark and N. Obers, *JHEP* **05**, 032 (2002), hep-th/0204047.
- [31] P. K. Townsend and M. Zamaklar, *Class. Quant. Grav.* **18**, 5269 (2001), hep-th/0107228.
- [32] J. H. Traschen, *Class. Quant. Grav.* **21**, 1343 (2004), hep-th/0308173.
- [33] T. Shiromizu, D. Ida, and S. Tomizawa, *Phys. Rev.* **D69**, 027503 (2004), gr-qc/0309061.
- [34] B. Kol, E. Sorkin, and T. Piran, *Phys. Rev.* **D69**, 064031 (2004), hep-th/0309190.
- [35] T. Harmark and N. A. Obers, *Class. Quant. Grav.* **21**, 1709 (2004), hep-th/0309116.
- [36] T. Harmark and N. A. Obers, *Nucl. Phys.* **B684**, 183 (2004), hep-th/0309230.
- [37] R. Emparan and H. Reall, *Phys. Rev. Lett.* **88**, 101101 (2002), hep-th/0110260.
- [38] S. Gubser, *Class. Quant. Grav.* **19**, 4825 (2002), hep-th/0110193.
- [39] E. Sorkin (2004), hep-th/0402216.
- [40] T. Wiseman, *Class. Quant. Grav.* **20**, 1137 (2003), hep-th/0209051.
- [41] H. Kudoh and T. Wiseman, *Prog. Theor. Phys.* **111**, 475 (2004), hep-th/0310104.
- [42] E. Sorkin, B. Kol, and T. Piran, *Phys. Rev.* **D69**, 064032 (2004), hep-th/0310096.
- [43] T. Harmark, *Phys. Rev.* **D69**, 104015 (2004), hep-th/0310259.
- [44] D. Gorbos and B. Kol (2004), hep-th/0406002.
- [45] R. Myers, *Phys. Rev.* **D35**, 455 (1987).
- [46] B. Kol (2002), hep-th/0206220.
- [47] T. Wiseman, *Class. Quant. Grav.* **20**, 1177 (2003), hep-th/0211028.
- [48] B. Kol and T. Wiseman, *Class. Quant. Grav.* **20**, 3493 (2003), hep-th/0304070.
- [49] H. Kudoh and T. Wiseman (To appear).
- [50] T. Hotta, J. Nishimura, and A. Tsuchiya, *Nucl. Phys.* **B545**, 543 (1999), hep-th/9811220.
- [51] W. Krauth and M. Staudacher, *Phys. Lett.* **B453**, 253 (1999), hep-th/9902113.
- [52] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, *Rev. Mod. Phys.* **53**, 43 (1981).
- [53] D. Kabat and G. Lifschytz, *Nucl. Phys.* **B571**, 419 (2000), hep-th/9910001.
- [54] A. Dumitru, Y. Hatta, J. Lenaghan, K. Orginos, and R. D. Pisarski (2003), hep-th/0311223.
- [55] J. M. Maldacena, *Phys. Rev. Lett.* **80**, 4859 (1998), hep-th/9803002.
- [56] S.-J. Rey and J.-T. Yee, *Eur. Phys. J.* **C22**, 379 (2001), hep-th/9803001.
- [57] J. Kiskis, R. Narayanan, and H. Neuberger, *Phys. Lett.* **B574**, 65 (2003), hep-lat/0308033.
- [58] R. Narayanan and H. Neuberger, *Phys. Rev. Lett.* **91**, 081601 (2003), hep-lat/0303023.
- [59] M. W. Choptuik et al., *Phys. Rev.* **D68**, 044001 (2003), gr-qc/0304085.
- [60] G. Horowitz and K. Maeda, *Phys. Rev. Lett.* **87**, 131301 (2001), hep-th/0105111.
- [61] G. Horowitz and K. Maeda, *Phys. Rev.* **D65**, 104028 (2002), hep-th/0201241.

- [62] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, Phys. Rev. **D55**, 5112 (1997), hep-th/9610043.
- [63] J. Polchinski, Prog. Theor. Phys. Suppl. **134**, 158 (1999), hep-th/9903165.
- [64] J. A. Harvey, G. W. Moore, and A. Strominger, Phys. Rev. **D52**, 7161 (1995), hep-th/9501022.
- [65] M. Bershadsky, A. Johansen, V. Sadov, and C. Vafa, Nucl. Phys. **B448**, 166 (1995), hep-th/9501096.
- [66] R. Dijkgraaf, E. Verlinde, and H. Verlinde, Nucl. Phys. **B500**, 43 (1997), hep-th/9703030.
- [67] L. Motl (1997), hep-th/9701025.
- [68] R. Dijkgraaf, J. M. Maldacena, G. W. Moore, and E. Verlinde (2000), hep-th/0005003.
- [69] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, Phys. Rept. **323**, 183 (2000), hep-th/9905111.
- [70] J. R. David, G. Mandal, and S. R. Wadia, Phys. Rept. **369**, 549 (2002), hep-th/0203048.
- [71] B. Sathiapalan, Mod. Phys. Lett. **A13**, 2085 (1998), hep-th/9805126.