Imperial College London

MSci EXAMINATION May 2013

This paper is also taken for the relevant Examination for the Associateship

GENERAL RELATIVITY

For 4th-Year Physics Students

XXX May 2013: 14:00 to 16:00

The paper consists of two sections: A and B
Section A contains one question, comprising small parts. [20 marks total]
Section B will contain four questions on selected parts of the course. [15 marks each]

Candidates are required to:
Answer **ALL** *parts of Section A and* **TWO** *questions from Section B.*

Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the 3 answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in 3 answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Conventions:

We use conventions as in lectures. In particular we take $(-,+,+,+)$ signature.

You may find the following formulae useful:

The Christoffel symbol is defined as,

$$\Gamma^{\mu}{}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta} \right)$$

The covariant derivative is given as,

$$\nabla_\mu v^\nu \equiv \partial_\mu v^\nu + \Gamma^{\nu}{}_{\mu\alpha} v^\alpha$$

The Riemann tensor is defined as,

$$R_{\alpha\beta\mu}{}^{\delta} = \partial_\beta \Gamma^{\delta}{}_{\alpha\mu} - \partial_\alpha \Gamma^{\delta}{}_{\beta\mu} + \Gamma^{\nu}{}_{\alpha\mu} \Gamma^{\delta}{}_{\beta\nu} - \Gamma^{\nu}{}_{\beta\mu} \Gamma^{\delta}{}_{\alpha\nu}$$

For a Lagrangian of a curve $x^\mu(\lambda)$ of the form,

$$F = \int d\lambda \, \mathcal{L}(x^\mu, \frac{dx^\mu}{d\lambda})$$

the Euler-Lagrange equations are,

$$\frac{d}{d\lambda} \left(\frac{\partial \mathcal{L}}{\partial (\frac{dx^\mu}{d\lambda})} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

Section A

Answer all of section A.

SECTION A

1. This question concerns accelerated motion.

(i) Suppose we have a particle following a trajectory $x^{\mu}(\tau)$ in a general spacetime.Here τ is the particle's proper time. The particle's 4-velocity v^{μ} is defined as $v^{\mu} = dx^{\mu}/d\tau$. Show that $v^{\mu}v_{\mu} = -1$.

ANSWER:

Testing material seen in lectures.

In an infinitesimal time $d\tau$ the spacetime interval will be,

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}d\tau^2 \tag{1}$$

Now proper time for a particle is defined to be $ds^2 = -d\tau^2$ and so,

$$-1 = g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = g_{\mu\nu}v^{\mu}v^{\nu} \tag{2}$$

Probable mark assignment:

[4 marks]

(ii) Show that the 4-velocity transforms as vector.

ANSWER:

Testing material seen in lectures.

Under a coordinate transform, so that $x'^{\mu'} = x'^{\mu'}(x^{\nu})$, then using the chain rule,

$$v'^{\mu'} = \frac{dx'^{\mu'}}{d\tau} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}}\frac{dx^{\mu}}{d\tau} = \frac{\partial x'^{\mu'}}{\partial x^{\mu}}v^{\mu} \tag{3}$$

and hence this does indeed transform as a vector.

Probable mark assignment:

[1 mark]

(iii) The 4-acceleration a^{μ} is defined as $a^{\mu} = v^{\nu}\nabla_{\nu}v^{\mu}$. Show that in Minkowski spacetime this reduces to $a^{\mu} = d^2x^{\mu}/d\tau^2$.

ANSWER:

Testing material given in lectures.

In Minkowski spacetime $\Gamma^{\alpha}{}_{\mu\nu} = 0$ and so,

$$a^{\mu} = v^{\nu}\nabla_{\nu}v^{\mu} = v^{\nu}\partial_{\nu}v^{\mu} = \frac{dx^{\nu}}{d\tau}\frac{\partial v^{\mu}}{\partial x^{\nu}} = \frac{dv^{\mu}}{d\tau} = \frac{d^2x^{\mu}}{d\tau^2} \tag{4}$$

Probable mark assignment:

[2 marks]

[This question continues on the next page . . .]

(iv) Show that a^μ and v^μ are orthogonal 4-vectors. Hence deduce that a^μ must be a spacelike vector.

ANSWER:

Testing material seen in lectures.

Start with $v^\mu v_\mu = -1$. Then act with $v^\nu \nabla_\nu$ to obtain,

$$0 = v^\nu \nabla_\nu \left(v^\mu v_\mu \right) = 2 v^\mu v^\nu \nabla_\nu v_\mu = 2 v^\mu a_\mu \tag{5}$$

Go to the instantaneous local inertial frame of the particle, so that at some point p on its trajectory then $v^\mu = (1, 0, 0, 0)$ and $g_{\mu\nu} = \eta_{\mu\nu}$ at that point. Then since $a^\mu v_\mu = 0$ then, $a^t = 0$, so that,

$$a^\mu = (0, a^i) \tag{6}$$

for 3-vector a^i, and so,

$$g_{\mu\nu} a^\mu a^\nu = \delta_{ij} a^i a^j = (a^1)^2 + (a^2)^2 + (a^3)^2 \geq 0 \tag{7}$$

at the point p. Note that if $g_{\mu\nu} a^\mu a^\nu = 0$ then $a^\mu = 0$ and the acceleration vanishes. So for non-vanishing acceleration $g_{\mu\nu} a^\mu a^\nu > 0$ at p and so a^μ is spacelike there. But we could have chosen p to be any point on the trajectory, and hence a^μ must always be spacelike.

Probable mark assignment:

[1 mark]

(v) Write down the geodesic equation for a timelike curve $x^\mu(\tau)$ parameterized by proper time? Hence show a non-accelerated particle follows a geodesic.

ANSWER:

Testing material seen in lectures.

The geodesic equation is,

$$v^\nu \nabla_\nu v^\mu = 0 \tag{8}$$

where $v^\mu = dx^\mu / d\tau$. Hence if $a^\mu = 0$, then the particle trajectory must obey the geodesic equation.

Probable mark assignment:

[1 mark]

(vi) Now consider a particle moving in the Schwarzschild spacetime, with coordinates $x^\mu = (t, r, \theta, \phi)$ and metric,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{9}$$

[This question continues on the next page . . .]

Consider a particle accelerating to stay at constant spatial position, so that r, θ, ϕ are all constant for the particle. Use the fact that,

$$\Gamma^r{}_{tt} = \frac{M}{r^3}\left(1 - \frac{2M}{r}\right), \Gamma^t{}_{tt} = \Gamma^\theta{}_{tt} = \Gamma^\phi{}_{tt} = 0 \tag{10}$$

Calculate the norm, $\sqrt{a^\mu a_\mu}$, of the spacelike 4-acceleration assuming that $r > 2M$. What happens to this norm at $r = 2M$ and why?

ANSWER:

Testing material given in lectures.

The 4-velocity is $v^\mu = (f, 0, 0, 0)$ for some function f since the particle is kept at fixed position in space. Then since $g_{\mu\nu}v^\mu v^\nu = -1$ then,

$$-\left(1 - \frac{2M}{r}\right)f^2 = -1 \tag{11}$$

so that,

$$f = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \tag{12}$$

The 4-acceleration is,

$$a^\mu = v^\nu \nabla_\nu v^\mu = v^\nu \partial_\nu v^\mu + v^\nu v^\alpha \Gamma^\mu{}_{\nu\alpha} = v^t \partial_t v^\mu + v^t v^t \Gamma^\mu{}_{tt} = f^2 \Gamma^\mu{}_{tt} = \frac{1}{1 - \frac{2M}{r}} \Gamma^\mu{}_{tt} \tag{13}$$

Using the Christoffel components given in the question we have, $a^t = a^\theta = a^\phi = 0$ and,

$$a^r = \frac{1}{1 - \frac{2M}{r}} \Gamma^r{}_{tt} = \frac{1}{1 - \frac{2M}{r}} \frac{M}{r^3}\left(1 - \frac{2M}{r}\right) = \frac{M}{r^3} \tag{14}$$

Then the norm,

$$a^\mu a_\mu = g_{rr}(a^r)^2 = \frac{1}{1 - \frac{2M}{r}} \frac{M^2}{r^6} \tag{15}$$

so that,

$$\sqrt{a^\mu a_\mu} = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \frac{M}{r^3} \tag{16}$$

This is indeed spacelike (ie. > 0) for $r > 2M$ and diverges, $\sqrt{a^\mu a_\mu} \to \infty$ at $r = 2M$. This is the horizon of the black hole, and an infinite acceleration is required to keep a timelike particle sitting at the horizon.

Probable mark assignment:

[2 marks]

[Total 11 marks]

Section B

Answer 2 out of the 4 questions in the following section.

SECTION B

2. This question concerns the Einstein equations for a star made of perfect fluid.

(i) Write down the stress tensor $T_{\mu\nu}$ for a perfect fluid in terms of the fluid energy density ρ, pressure P and 4-velocity u^μ (recall $u^\mu u_\mu = -1$). Consider n^μ to be a vector field that is orthogonal to v^μ, so $v^\mu n_\mu = 0$. Use stress energy conservation, and consider the quantity $n^\mu \nabla^\nu T_{\mu\nu}$ to derive one of the fluid equations,

$$n^\mu \left(\partial_\mu P + (\rho + P)\, u^\nu \nabla_\nu u_\mu \right) = 0 \tag{1}$$

ANSWER:

Similar problem ...

The stress tensor is;

$$T_{\mu\nu} = \rho u_\mu u_\nu + P \left(u_\mu u_\nu + g_{\mu\nu} \right) \tag{2}$$

Conservation is;

$$\begin{aligned} 0 = \nabla^\mu T_{\mu\nu} &= (\nabla^\mu \rho) u_\mu u_\nu + \rho \left(u_\mu \nabla^\mu u_\nu + u_\nu \nabla^\mu u_\mu \right) \\ &\quad + (\nabla^\mu P) \left(u_\mu u_\nu + g_{\mu\nu} \right) + P \left(u_\mu \nabla^\mu u_\nu + u_\nu \nabla^\mu u_\mu \right) \end{aligned} \tag{3}$$

where we recall $\nabla^\mu g_{\mu\nu} = 0$. Then contracting with n^ν and using $n^\nu u_\nu = 0$ gives,

$$\begin{aligned} 0 = u^\nu \nabla^\mu T_{\mu\nu} &= \rho \left(n^\nu u_\mu \nabla^\mu u_\nu \right) + (\nabla^\mu P) \left(n^\nu g_{\mu\nu} \right) + P \left(n^\nu u_\mu \nabla^\mu u_\nu \right) \\ &= n^\nu (\nabla_\mu P) + n^\nu \left(\rho + P \right) \left(u_\mu \nabla^\mu u_\nu \right) \end{aligned} \tag{4}$$

and $\nabla_\mu P = \partial_\mu P$ as it is a scalar, and hence this gives the result.

Probable mark assignment: 1 mark for

[1 mark]

(ii) Consider a static (ie. time independent) metric describing a spherically symmetric star. We take coordinates $x^\mu = (t, r, \theta, \phi)$ and a metric,

$$ds^2 = -e^{2f(r)} dt^2 + \left(1 - \frac{m(r)}{r} \right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \tag{5}$$

where $f(r)$ and $m(r)$ are functions of the radial coordinate r. Consider this to be the solution to the Einstein equations for perfect fluid, where the fluid is static so that $u^\mu = (T(r), 0, 0, 0)$. Firstly determine the function $T(r)$. Then using your

[This question continues on the next page . . .]

answer to part i) above, choosing $n^\mu = (0, 1, 0, 0)$ and computing the necessary components of $\Gamma^{\alpha}{}_{\mu\nu}$, show that,

$$\frac{dP}{dr} = -(\rho + P)\frac{df}{dr} \tag{6}$$

ANSWER:

Similar problem ...

Now since $g_{\mu\nu}u^\mu u^\nu = -1$ we have,

$$-1 = g_{\mu\nu}u^\mu u^\nu = g_{tt}u^t u^t = -e^{2f(r)}T^2 \tag{7}$$

and so,

$$T = e^{-f(r)} \tag{8}$$

Consider the equation from part i) with $n^\mu = (0, 1, 0, 0)$, then,

$$\begin{aligned} 0 &= n^\mu \left(\partial_\mu P + (\rho + P)\, u^\nu \nabla_\nu u_\mu\right) \\ &= \partial_r P + (\rho + P)\, u^\nu \nabla_\nu u_r \\ &= \partial_r P + (\rho + P)\left(u^\nu \partial_\nu u_r + \Gamma^{\alpha}{}_{\nu r} u^\nu u_\alpha\right) \\ &= \partial_r P + (\rho + P)\left(u^t \partial_t u_r + \Gamma^{t}{}_{tr} u^t u_t\right) \\ &= \partial_r P + (\rho + P)\,\Gamma^{t}{}_{tr} g_{tt} u^t u^t \\ &= \partial_r P + (\rho + P)\,\Gamma^{t}{}_{tr} e^{2f(r)} T^2 \\ &= \partial_r P + (\rho + P)\,\Gamma^{t}{}_{tr} \end{aligned} \tag{9}$$

Now we require $\Gamma^{t}{}_{tr}$;

$$\begin{aligned} \Gamma^{t}{}_{tr} &= \frac{1}{2} g^{t\nu}\left(\partial_t g_{r\nu} + \partial_r g_{t\nu} - \partial_\nu g_{tr}\right) \\ &= \frac{1}{2} g^{tt}\left(\partial_t g_{rt} + \partial_r g_{tt} - \partial_t g_{tr}\right) \\ &= \frac{1}{2} g^{tt} \partial_r g_{tt} \\ &= \frac{1}{2} e^{-2f(r)} \partial_r e^{2f(r)} \\ &= \partial_r f \end{aligned} \tag{10}$$

and then,

$$\begin{aligned} 0 &= \partial_r P + (\rho + P)\,\Gamma^{t}{}_{tr} \\ &= \partial_r P + (\rho + P)\,\partial_r f \end{aligned} \tag{11}$$

as required.

[This question continues on the next page . . .]

Probable mark assignment: 1 mark for

[1 mark]

(iii) Now the non-zero components of the Ricci tensor with one index up and one down, $R^{\mu}{}_{\nu}$, are,

$$\begin{aligned}
R^{t}{}_{t} &= -f''(r)\left(1-\frac{m(r)}{r}\right)+f'(r)\left(\frac{rm'(r)+3m(r)-4r}{2r^2}\right)-f'(r)^2\left(1-\frac{m(r)}{r}\right) \\
R^{r}{}_{r} &= \frac{rm'(r)-m(r)}{r^3}-f''(r)\left(1-\frac{m(r)}{r}\right)+f'(r)\left(\frac{rm'(r)-m(r)}{2r^2}\right)-f'(r)^2\left(1-\frac{m(r)}{r}\right) \\
R^{\theta}{}_{\theta} &= R^{\phi}{}_{\phi}=\frac{m(r)+rm'(r)}{2r^3}+f'(r)\left(\frac{m(r)-r}{r^2}\right) \qquad (12)
\end{aligned}$$

Use these to calculate the Einstein tensor components, G_{tt} and G_{rr}, and hence show the Einstein equations imply,

$$m(r)=8\pi G_N\int_0^r \rho(r')r'^2dr' \ , \qquad \frac{df}{dr}=\frac{m(r)+8\pi G_N r^3P(r)}{2r\left(r-m(r)\right)} \qquad (13)$$

ANSWER:

Similar problem ...

The Ricci scalar is,

$$\begin{aligned}
R &= R^{t}{}_{t}+R^{r}{}_{r}+R^{\theta}{}_{\theta}+R^{\phi}{}_{\phi} \\
&= R^{t}{}_{t}+R^{r}{}_{r}+2R^{\theta}{}_{\theta} \\
&= 2\frac{m'(r)}{r^2}-2f''(r)\left(1-\frac{m(r)}{r}\right)+f'(r)\left(\frac{rm'(r)+3m(r)-4r}{r^2}\right)-2f'(r)^2\left(1-\frac{m(r)}{r}\right)
\end{aligned}$$

Then,

$$\begin{aligned}
G_{tt} &= R_{tt}-\frac{1}{2}g_{tt}R \\
&= g_{tt}R^{t}{}_{t}-\frac{1}{2}g_{tt}R \\
&= g_{tt}\left(R^{t}{}_{t}-\frac{1}{2}R\right) \qquad (14)
\end{aligned}$$

where,

$$R^{t}{}_{t}-\frac{1}{2}R = -\frac{m'(r)}{r^2} \qquad (15)$$

so,

$$G_{tt} = e^{2f(r)}\frac{m'(r)}{r^2} \qquad (16)$$

[This question continues on the next page . . .]

And,

$$\begin{aligned} G_{rr} &= R_{rr} - \frac{1}{2} g_{rr} R \\ &= g_{rr} R^{r}{}_{r} - \frac{1}{2} g_{rr} R \\ &= g_{rr} \left(R^{r}{}_{r} - \frac{1}{2} R \right) \end{aligned} \tag{17}$$

where,

$$R^{r}{}_{r} - \frac{1}{2} R = -\frac{m(r)}{r^3} + 2f'(r) \left(\frac{r - m(r)}{r^2} \right) \tag{18}$$

so,

$$\begin{aligned} G_{rr} &= \frac{1}{1 - \frac{m(r)}{r}} \left(-\frac{m(r)}{r^3} + 2f'(r) \left(\frac{r - m(r)}{r^2} \right) \right) \\ &= -\frac{m(r)}{r^3 - m(r) r^2} + \frac{2}{r} f'(r) \end{aligned} \tag{19}$$

The Einstein equation is,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu} \tag{20}$$

Now,

$$T_{tt} = \rho u_t^2 + P \left(u_t u_t + g_{tt} \right) \tag{21}$$

and $u_t = g_{tt} u^t = -e^{2f(r)} T = -e^{f(r)}$, so,

$$T_{tt} = \rho e^{2f(r)} \tag{22}$$

and,

$$T_{rr} = \rho u_r^2 + P \left(u_r u_r + g_{rr} \right) = \frac{P}{1 - \frac{m(r)}{r}} \tag{23}$$

Then the tt component of the Einstein equation is;

$$-e^{2f(r)} \frac{m'(r)}{r^2} = 8\pi G_N \left(\rho e^{2f(r)} \right) \tag{24}$$

so we find,

$$\frac{m'(r)}{r^2} = 8\pi G_N \rho \tag{25}$$

[This question continues on the next page . . .]

so we find as required,

$$m(r) = 8\pi G_N \int_0^r dr' \rho(r') r'^2 \tag{26}$$

And for the rr component,

$$-\frac{m(r)}{r^3 - m(r) r^2} + \frac{2}{r} f'(r) = 8\pi G_N \left(\frac{P}{1 - \frac{m(r)}{r}} \right) \tag{27}$$

so,

$$f'(r) = \frac{1}{2r} \left(\frac{m(r) + 8\pi G_N r^3 P}{r - m(r)} \right) \tag{28}$$

Probable mark assignment: 1 mark for

[1 mark]

(iv) If the star has a surface, say at $r = R$, then outside this surface for $r > R$ there is no fluid matter ie. $\rho = P = 0$. Then solve the equations to determine the metric in the star's exterior. What is this exterior spacetime? What is the mass of the star in terms of the function $m(r)$?

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[Total 4 marks]

3. This question concerns scalar fields and FLRW spacetime.

(i) Consider a scalar field $\phi(t, x^i)$ with potential $V(\phi)$ on a general spacetime. Its stress tensor is given as,

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \left(\nabla^\alpha \phi \nabla_\alpha \phi + V(\phi) \right) \tag{1}$$

Use stress energy conservation to determine the equation of motion of this scale field.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(ii) Now take the spacetime to be FLRW, with coordinates $x^\mu = (t, x^i)$, and metric,

$$ds^2 = -dt^2 + a(t)^2 dx^i dx^i \tag{2}$$

Compute the Christoffel symbol components $\Gamma^\alpha{}_{\mu\nu}$ for this metric.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(iii) Take the scalar to have the symmetries of FLRW, so that ϕ is only a function of time t. Also take its potential to vanish, $V(\phi) = 0$ - this is a *massless* scalar field. Show that the change in the scalar at time t from some initial time t_0 is given as,

$$\phi(t) - \phi(t_0) = \int_{t_0}^{t} dt' a(t')^3 \tag{3}$$

Consider the stress tensor for this massless scalar field in FLRW. By computing the components of $T_{\mu\nu}$ explicitly, show that it behaves in the same way as a perfect fluid with equation of state $\rho = P$.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[This question continues on the next page . . .]

(iv) Consider now the opposite limit of the scalar field behaviour, where we can ignore the scalar field time dependence so $\phi'(t)^2 \ll V(\phi)$ - this is a *potential dominated* scalar field. Use the fact that the *tt*-component of the Einstein tensor is given as,

$$G_{tt} = 3\left(\frac{a'(t)}{a(t)}\right)^2 \tag{4}$$

and the scalar is approximately independent of time to show the scale factor may expand exponentially quickly in time as,

$$a(t) = \exp\left(t\sqrt{\frac{4\pi}{3}G_N V(\phi)}\right) \tag{5}$$

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[Total 4 marks]

4. This question concerns Nordström's theory of gravity.

(i) Before Einstein completed his equations of General Relativity, an alternative theory was proposed by Nordström. As with Einstein's theory, in Nordström's theory gravity is due to curvature of spacetime. However, the theory is much simpler as the spacetime metric cannot be general, but is given in terms of one function $\Phi(t, x^i)$, as,

$$ds^2 = \Phi^2 \left(-dt^2 + dx^i dx^i \right) \tag{1}$$

where we take coordinates $x^\mu = (t, x^i)$. Nordström proposed the gravitational field equation is,

$$R = \kappa T \tag{2}$$

where R is the Ricci scalar, and T is the trace of the stress tensor, and κ is a constant.

Show that,

$$\Gamma^\alpha{}_{\mu\nu} = \tag{3}$$

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(ii) Now compute the Ricci tensor, and show that the field equation yields,

$$\frac{6}{\Phi^3} \nabla^\mu \nabla_\mu \Phi = -\kappa T \tag{4}$$

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(iii) A massive particle in the spacetime follows the timelike curve $x^\mu = (T(\tau), X^i(\tau))$ where τ is its proper time. Assume the Nordstöm scalar Φ is time independent, so $\Phi = \Phi(x^i)$. Use the Euler-Lagrange equations to vary the Lagrangian,

$$L = \int d\tau\, \Phi^2(X) \left(-\left(\frac{dT}{d\tau} \right)^2 + \frac{dX^i}{d\tau} \frac{dX^i}{d\tau} \right) \tag{5}$$

[This question continues on the next page . . .]

and hence determine the equation of the geodesic curves.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(iv) Consider a Newtonian limit similar to that in GR by taking $\Phi = 1 + \epsilon\phi + O(\epsilon^{3/2})$. Use you answer to part iii) to identify ϕ with the usual Newton gravitational potential. Consider slowly moving dust fluid, and show that the constant κ should be $\kappa = 24\pi G_N$.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[Total 4 marks]

5. This question concerns the Newtonian limit and light bending. Recall the metric for the Newtonian spacetime is;

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \ , \quad \text{with} \quad g_{\mu\nu} = \eta_{\mu\nu} - 2\epsilon\, \Phi(x^i)\, \delta_{\mu\nu} + O(\epsilon^{3/2})$$

where $x^\mu = (t, x^i)$ with $i = 1, 2, 3$ and we will take the potential $\Phi(x^i)$ to be static so $\partial_t \Phi = 0$.

(i) Take a massive particle moving on the curve, $x^\mu(\tau)$, where τ is its proper time. Assume the motion is *slow* so that,

$$\frac{dx^\mu}{d\tau} = \left(1 + \epsilon\, f + O(\epsilon^{3/2}),\ \epsilon^{1/2}\, v^i + O(\epsilon^{3/2})\right)$$

Determine the function f in terms of the 3-velocity v^i and potential Φ. Suppose the particle is moving through a curved region of space where $\Phi \neq 0$, and we are sitting very far away at constant x^i in a region where $\Phi = 0$. If the particle emitted light with frequency ω, what frequency do we observe the light to have?

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

(ii) Parameterize a null curve $x^\mu(\lambda) = (T(\lambda), X^i(\lambda))$ by an affine parameter λ. By varying the Lagrangian,

$$L = \int d\lambda g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \qquad (1)$$

show that in the Newtonian limit the equations governing a light ray are;

$$\left(\frac{dT}{d\lambda}\right)^2 = \frac{dX^i}{d\lambda}\frac{dX^i}{d\lambda} \ , \quad \frac{dT}{d\lambda} = k\,(1 - \epsilon\Phi) + O(\epsilon^{3/2}) \ , \quad \frac{d^2 X^i}{d\lambda^2} = \epsilon \partial_j \Phi \frac{dX^i}{d\lambda}\frac{dX^j}{d\lambda} + O(\epsilon^{3/2}) \quad (2)$$

for some constant k.

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[This question continues on the next page . . .]

(iii) Take the Newtonian potential for a static point source with mass M at position $x^i = (0, R, 0)$. Consider a light ray initially propagating along the x^1 axis, so that $x^\mu = (\lambda, \lambda, 0, 0)$ for $\lambda \to -\infty$. The trajectory of the ray is then

$$x^\mu = (\lambda + \epsilon F(\lambda) + O(\epsilon^{3/2})), \lambda + \epsilon G(\lambda) + O(\epsilon^{3/2})), \epsilon H(\lambda) + O(\epsilon^{3/2})), 0) \qquad (3)$$

Determine $H(\lambda)$, and hence show that light is deflected by an angle θ, where,

$$\theta = \epsilon \frac{G_N M}{4R} + O(\epsilon^{3/2}) \qquad (4)$$

ANSWER:

Similar problem ...

Probable mark assignment: 1 mark for

[1 mark]

[Total 3 marks]